ON THE CONVERGENCE OF A SERIES OF BUNDSCHUH

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ABSTRACT. If \((c_n)_{n\geq 1}\) is a non-increasing sequence of positive real numbers tending to 0 in P. Bundschuh [Arch. Math. 29 (1977), 518–523] the question was asked for which real numbers \(\alpha\) the series \(\sum_{n=1}^{\infty} (-1)^{[2^{2n}\alpha]} c_n\) is convergent. For series of the form \(g(\alpha) = \sum_{n=1}^{\infty} (-1)^{[2^{2n}\alpha]}\) it has been shown in Bundschuh [ibid] that it converges for numbers \(\alpha\) with bounded continued fraction expansion and also for numbers like \(e\), thereby solving a problem posed in H.D. Ruderman [Amer. Math. Monthly 83 (1977), 573]. Whether the series is convergent for \(\alpha = \pi\) has remained open. We could also deal with the more general series \(\sum_{n=1}^{\infty} (-1)^{[2^{2n}\alpha]} c_n\), but then the technique would obscure the ideas of the proof.

The author and S. Tříčková [J. Math. Anal. Appl. 342 (2006), 238–247] have proved that \(\sum_{n=1}^{\infty} (-1)^{[2^{2n}\alpha]} n\) is convergent almost everywhere, that the function \(g(\alpha)\) is odd, has period 1 and represents a function in \(L^2[0,1]\). Furthermore every open non empty interval contains two subsets \(P\) and \(N\) of the power of the continuum such that for \(\alpha \in P\) we have \(g(\alpha) = +\infty\) and for \(\alpha \in N\) we have \(g(\alpha) = -\infty\). If \(\alpha = \frac{p}{q}\) is rational and \(p\) and \(q\) are coprime then the series is convergent if and only if \(q\) is even.

In this paper we determine all real numbers \(\alpha\) for which the series \(g(\alpha)\) is convergent.

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Statement and proof of the result

In the sequel we may assume that \(\alpha\) is an irrational number and that \(0 < \alpha < \frac{1}{2}\). Let \(\alpha = [0; a_1, a_2, \ldots]\) be the continued fraction expansion of \(\alpha\) with convergents

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easily be shown that expansion of \( x = b \) (with \( N \))

\[ s_{i,j} = q_{\min(i,j)}(q_{\max(i,j)} - p_{\max(i,j)}) \]  

By \( \{t\} \) we denote the fractional part of the real number \( t \).

If \( N \geq 1 \) is a positive integer there are uniquely determined non-negative integers \( b_0, b_1, \ldots \) such that \( b_0 < a_1, b_1 \leq a_{i+1} b_i = a_{i+1} \implies b_i = 0 \), and such that \( N = \sum_{i=0}^{\infty} b_i q_i \). This representation is called the Ostrowski-expansion of \( N \) to base \( \alpha \). There is clearly a largest \( m \geq 0 \) such that \( b_m \neq 0 \).

For \( 0 < x < 1 \) there are uniquely determined non-negative integers \( c_0, c_1, \ldots \) and \( \delta \in \{0,1\} \) such that \( c_0 < a_1, c_i \leq a_{i+1} c_i = a_{i+1} \implies c_i = 0 \), and

\[ x = \sum_{i=0}^{\infty} c_i (q_i \alpha - p_i) + \delta. \]

This representation is called the Ostrowski-expansion of \( x \) to base \( \alpha \). By \( i_x \) we denote the smallest \( i \) such that \( c_i \neq 0 \). It can easily be shown that \( \delta = 1 \iff i_x \) is odd \( \iff x > 1 - \alpha \).

By \( c_{0,x} \) we denote the characteristic function of the interval \([0, x)\). The following Theorem can be found in [3] or [6, p.361].

**Theorem A.** Let \( N \) be a positive integer with Ostrowski-expansion \( \sum_{n=1}^{m} b_n q_i \) (with \( b_m \neq 0 \)), and let \( x \) be in \((0,1)\) with Ostrowski-expansion

\[ x = \sum_{i=0}^{\infty} c_i (q_i \alpha - p_i) + \delta, \]

both to base \( \alpha \). Assume that \( x \notin \{ \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}\} \). For \( k \geq -1 \) let

\[ \delta_k = \begin{cases} 1 & \sum_{i=0}^{k} c_i q_i > \sum_{i=0}^{k} b_i q_i \\ 0 & \text{else} \end{cases} \]

and let \( t \) be the first integer \( \geq m \) with \( \delta_t = 1 \).

Then

\[ \sum_{n=1}^{N} c_{0,x}(\{n\alpha\}) - Nx = \sum_{i=0}^{\infty} (-1)^i \min\{c_i, b_i\} - \sum_{i=0}^{m} (-1)^i \delta_i \delta_{i-1} \]

\[ -\sum_{i=0}^{\infty} \sum_{j=0}^{i} b_j c_j s_{i,j} - \frac{1}{2} \left((-1)^{x} - (-1)^{t}\right). \]

**Lemma 1.** For \( i \geq 0 \) let \( c_i = \frac{1}{4} (a_{i+1} + \frac{1}{2}(-1)^{i} - \frac{1}{2}(-1)^{i+1}) (1 - (-1)^{q_0}). \)

Then for \( j \geq 0 \) we have \( \sum_{i=0}^{\infty} c_i s_{i,j} = \frac{1}{4}(-1)^{j} (1 - (-1)^{q_0}). \)
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Proof. We note that \((-1)^{q_{i+1}} - (-1)^{q_i}(1 + (-1)^{q_i}) = 0\). Therefore as 

\[ a_{i+1}s_{i,j} = s_{i+1,j} - s_{i-1,j} + (-1)^j \delta_{i,j} \]

we have

\[
\sum_{i=0}^{\infty} c_is_{i,j} = \frac{1}{4} \sum_{i=0}^{\infty} (s_{i+1,j} - s_{i-1,j} + (-1)^j \delta_{i,j})(1 - (-1)^{q_i})
+ \frac{1}{8} \sum_{i=0}^{\infty} s_{i,j}((-1)^{q_{i-1}} - (-1)^{q_{i+1}})(1 - (-1)^{q_i})
= \frac{1}{4} \sum_{i=0}^{\infty} s_{i,j}(1 - (-1)^{q_{i-1}}) - \frac{1}{4} \sum_{i=0}^{\infty} s_{i,j}(1 - (-1)^{q_{i+1}})
+ \frac{1}{4} (-1)^j(1 - (-1)^{q_i}) + \frac{1}{8} \sum_{i=0}^{\infty} s_{i,j}((-1)^{q_{i-1}} - (-1)^{q_{i+1}})(1 - (-1)^{q_i})
= \frac{1}{4} (-1)^j(1 - (-1)^{q_i}) + \frac{1}{8} \sum_{i=0}^{\infty} s_{i,j}((-1)^{q_{i+1}} - (-1)^{q_{i-1}})(1 + (-1)^{q_i})
= \frac{1}{4} (-1)^j(1 - (-1)^{q_i}).
\]

By a similar computation one can show that

\[
\sum_{i=0}^{m} c_iq_i = \frac{1}{4} q_{m+1}(1 - (-1)^{q_m}) + \frac{1}{4} q_m(1 - (-1)^{q_{m+1}}).
\]

Lemma 2. For \(i \geq 0\) let \(c_i := \frac{1}{4} (a_{i+1} + \frac{1}{2} (-1)^{q_i-1} - \frac{1}{2} (-1)^{q_{i+1}})(1 - (-1)^{q_i}).\)
Then \(\sum_{i=0}^{\infty} c_i(q_i \alpha - p_i)\) is the Ostrowski-expansion of \(\frac{1}{4}\) to base \(\alpha\).

Proof. First we prove that \(c_i\) is a non-negative integer. This is clear if \(2|q_i\).
Otherwise

\[ c_i = \frac{1}{2} (a_{i+1} + \frac{1}{2} (-1)^{q_i-1} - \frac{1}{2} (-1)^{q_{i+1}}). \]

If \(a_{i+1}\) is even, then \(q_{i-1} \equiv q_{i+1} \pmod{2}\) and we are done again. Otherwise \(q_{i-1} \not\equiv q_{i+1} \pmod{2}\), and therefore again \(0 \leq c_i\), and \(c_i \in \mathbb{Z}\).
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As \( \alpha < \frac{1}{2} \) we have \( a_1 > 1 \). Note that therefore \( c_0 = \frac{1}{2}(a_1 + \frac{1}{2} - \frac{1}{2}(-1)^{a_1}) < a_1 \).
If \( c_i = a_{i+1} \), then \( 2 \mid q_i \) and \( a_{i+1} = \frac{1}{2}(a_i + 1) - \frac{1}{2}(-1)^{q_i} \), which implies \( a_{i+1} = 1 \) and that \( q_{i-1} \) is even. But then \( c_{i-1} = 0 \).

The representation \( \sum_{i=0}^{\infty} c_i(q_i\alpha - p_i) = \frac{1}{2} \) is a special case of Lemma 1 (put \( j = 0 \)).

**Corollary 1.** Let \( N \) be a positive integer with Ostrowski-expansion \( \sum_{n=1}^{m} b_n q_i \) (with \( b_m \neq 0 \)) to base \( \alpha \). For \( k \geq 0 \) let
\[
c_k = \frac{1}{2}(a_{k+1} + \frac{1}{2}(-1)^{q_i} - \frac{1}{2}(-1)^{q_{i+1}})(1 - (-1)^{q_i}),
\]
for \( k \geq -1 \) let
\[
\delta_k = \begin{cases} 1 & \text{if } \sum_{i=0}^{k} c_i q_i > \sum_{i=0}^{k} b_i q_i, \\ 0 & \text{else}, \end{cases}
\]
and let \( t \) be the first integer \( \geq m \) with \( \delta_t = 1 \). Then for \( \alpha < 1/2 \)
\[
\sum_{n=1}^{N} c([n\alpha]) - N/2 = \sum_{i=0}^{m} (-1)^i \left( \min\{c_i, b_i\} - \frac{1}{2} b_i (1 - (-1)^{q_i}) \right) - \sum_{i=0}^{m} (-1)^i \delta_i \delta_{i-1} - \frac{1}{2} (1 - (-1)^{q_i}) = \frac{1}{2} \sum_{i=0}^{m} (-1)^i \min\{b_i, a_{i+1} - b_i\} + O(m).
\]

**Proof.** This follows from Theorem A, Lemma 1 and Lemma 2 by noting that \( i_{t/2} = 0 \).

**Theorem 1.** Let \( \alpha \) be irrational with continued fraction expansion \( \alpha = [0; a_1, a_2, ...] \) and convergents \( \frac{b_n}{q_n} \). The following assertions are equivalent:

(1) \( \sum_{n=0}^{\infty} (-1)^{2n\alpha} \) is convergent (resp. \( = \infty \), resp. \( = -\infty \)).

(2) \( \sum_{t=0}^{\infty} (-1)^{t/2} \log a_{t+1} \) is convergent (resp. \( = \infty \), resp. \( = -\infty \)).
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Proof. First we restrict ourselves to the case $\alpha < 1/2$. Note that $\lfloor 2n\alpha \rfloor$ is even if and only if \{n\alpha\} < 1/2. Hence

$$\sum_{i=1}^{n} (-1)^{\lfloor 2i\alpha \rfloor} = \sum_{i=1}^{n} \left( c_{\lfloor 0,1/2 \rfloor}(\{i\alpha\}) - c_{\lfloor 1/2,1 \rfloor}(\{i\alpha\}) \right) = 2\left(\sum_{i=1}^{n} c_{\lfloor 0,1/2 \rfloor}(\{i\alpha\}) - n/2\right) = \sum_{j=0}^{\infty} (-1)^{j} \min\{b_{j}(n), a_{j+1} - b_{j}(n)\} + O(\log n),$$

by Corollary 1, as $q_{m} \leq n$ implies $m = O(\log n)$; here and in the sequel $b_{i}(n)$ denotes the $i$-th digit of $n$ in the Ostrowski-expansion of $n$ to base $\alpha$.

Therefore by summation by parts

$$\sum_{n=1}^{N} \frac{(-1)^{\lfloor 2na \rfloor}}{n} = \sum_{n=1}^{N-1} \frac{1}{n(n+1)} \left( \sum_{i=0}^{\infty} (-1)^{i} \min\{b_{i}(n), a_{i+1} - b_{i}(n)\} + O(\log n) \right) + \frac{2}{N} \left( \sum_{n=1}^{N} c_{\lfloor 0,1/2 \rfloor}(\{n\alpha\}) - N/2 \right).$$

The last term tends to 0 as $(n\alpha)_{n \geq 1}$ is uniformly distributed mod 1.

In order to simplify notation and to prove all three statements simultaneously we write for two sequences $(u_{n})_{n \geq 1}$, and $(v_{n})_{n \geq 1}$ $u_{n} \sim v_{n}$ if $u_{n} - v_{n}$ is convergent. Let $m$ be chosen such that $q_{m} \leq N < q_{m+1}$.

Then

$$\sum_{n=1}^{N} \frac{(-1)^{\lfloor 2na \rfloor}}{n} \sim \sum_{n=1}^{N} \frac{1}{n(n+1)} \sum_{i=0}^{\infty} (-1)^{i} \min\{b_{i}(n), a_{i+1} - b_{i}(n)\}.$$

The inner sum terminates. If $t_{n}$ is the largest index $t$ with $b_{t}(n) \neq 0$ (that is $q_{t} \leq n < q_{t+1}$) we have

$$\sum_{i<t_{n}} (-1)^{i} \min\{b_{i}(n), a_{i+1} - b_{i}(n)\} = O\left(\sum_{i<t_{n}} a_{i+1}\right)$$
and hence

\[ \sum_{n=1}^{N} \frac{1}{n(n+1)} \sum_{i<t} (-1)^i \min \{ b_i(n), a_{i+1} - b_i(n) \} = O(\sum_{t=0}^{m} \sum_{n=q_t}^{q_{t+1}-1} \frac{1}{n^2} \sum_{i<t} a_{i+1}) \]

\[ = O\left( \sum_{i=2}^{m} \frac{1}{q_i} \sum_{i<t} a_{i+1} \right) = O\left( \sum_{i=0}^{n} \frac{1}{t+1} \sum_{i<t} a_{i+1} \right) = O(1). \]

This implies

\[ \sum_{n=1}^{N} \frac{(-1)^{[2n\alpha]}}{n} \sim \sum_{n=1}^{N} \frac{1}{n(n+1)} (-1)^{t_n} \min \{ b_{t_n}(n), a_{t_n+1} - b_{t_n}(n) \} \]

\[ = \sum_{t=0}^{m} (-1)^t \sum_{n=q_t}^{q_{t+1}-1} \frac{1}{n(n+1)} \min \{ b_t(n), a_{t+1} - b_t(n) \} \]

\[ + \frac{1}{2} \left( 1 - (-1)^{q_m} \right) \sum_{n=q_m}^{N} (-1)^{m} \frac{1}{n(n+1)} \min \{ b_m(n), a_{m+1} - b_m(n) \}. \]

For \( q_t \leq n < q_{t+1} \) we have \( b_t(n) = \lfloor \frac{n}{q_t} \rfloor \). Therefore the first inner sum is equal to

\[ \sum_{q_t \leq n < q_{t+1}/2} \frac{1}{n(n+1)} \frac{n}{q_t} + \sum_{q_t \leq n < q_{t+1}/2} \frac{1}{n(n+1)} \left( a_{t+1} - \frac{n}{q_t} \right) + O\left( \frac{1}{q_t} \right) \]

\[ = \frac{1}{q_t} \log \frac{a_{t+1}}{2} + O\left( \frac{a_{t+1}}{n^2} \right) + O\left( \frac{1}{q_t} \right) \]

\[ = \frac{1}{q_t} \log a_{t+1} + O\left( \frac{1}{q_t} \right). \]

In the same way one sees that the last sum = \((-1)^m \frac{1}{q_m} \log \frac{N}{q_m} + O(\frac{1}{q_m})\). Collecting everything we get the result in this case.

If \( \alpha > 1/2 \), let \( 1 - \alpha = [0; a'_1, a'_2, \ldots] \), and let \( \frac{a'_i}{q'_m} \) be the convergents of \( 1 - \alpha \). Then \( a'_1 = a_2 + 1 \), \( a'_i = a_{i+1} \) for \( i > 1 \), \( q'_i = q_{i+1} \). From \( g(\alpha) = -g(1 - \alpha) \) we get the result in this case.

It is now clear that the series is convergent for \( \alpha = \pi \) as we know that then \( a_{t+1} = O(q'_t) \) for some absolute constant \( c > 0 \) (see [2]).
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