WEIGHTED SUMS IN FINITE ABELIAN GROUPS


ABSTRACT. In this note we prove the following weighted generalization of Bollobás and Leader theorem (J. Number Theory 78 (1999), no. 1, 27–35): Let \( G \) be an abelian group of order \( n \) and \( k \) a positive integer. Let \( (w_1, w_2, \ldots, w_k) \) be a sequence of integers where each \( w_i \) is co-prime to \( n \). Then, given a sequence \( (x_1, x_2, \ldots, x_{k+r}) \) of elements of \( G \), where \( 1 \leq r \leq n-1 \), if \( 0 \) is the most repeated element in the sequence, and \( \sum_1^k w_i x_{\sigma(i)} \neq 0 \), for all permutations \( \sigma \) of \( \{1, 2, \ldots, k+r\} \), we have

\[
\left| \left\{ \sum_1^k w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } \{1, 2, \ldots, k+r\} \right\} \right| \geq r+1.
\]

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1. Introduction

Let \( G \) be a finite abelian group (written additively) of order \( n \). For positive integers \( t, d \), with \( t \geq d \), given a sequence \( (a_1, a_2, \cdots, a_t) \) of length \( t \), by a \( d \)-sum of the sequence one means a sum \( a_{i_1} + \cdots + a_{i_d} \) of the elements in a subsequence of length \( d \).

A result of Bollobás–Leader [2] is the following:

**Theorem A.** Suppose we are given an abelian group \( G \) of order \( n \) and a sequence \( (a_1, a_2, \cdots, a_{n+r}) \) of elements of \( G \), where \( r \) is a positive integer. Then, if \( 0 \) is not an \( n \)-sum, the number of distinct \( n \)-sums of the sequence is at least \( r+1 \).

For \( n \geq 3 \), taking \( r = n - 2 \) in the above result, one observes that given a sequence of length \( 2n - 2 \) of elements of \( G \), if \( 0 \) is not an \( n \)-sum, then the set of \( n \)-sums is the set \( G \setminus \{0\} \), that is, any non-zero element of \( G \) is an \( n \)-sum.

Similarly, for \( n \geq 2 \), taking \( r = n - 1 \) in Theorem A, it follows that given a sequence of length \( 2n - 1 \) of elements of \( G \), \( 0 \) must be an \( n \)-sum. When \( G \) is

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cyclic, this is the content of the well known theorem of Erdős–Ginzburg–Ziv [4] (can also see [1] or [8], for instance).

In the present paper, following the method of a simple proof of the above result of Bollobás and Leader as given by Yu [10], we prove a theorem which will imply (see Corollary 2) a result of Hamidoune [6], which had confirmed a conjecture of Caro [3] (see also [5], for instance) in a special case. For further information regarding these results, we refer to the paper of Grynkiewicz [5], where, among other things, the above mentioned conjecture of Caro has been established in full generality.

In what follows, we shall use the following notations. For a positive integer \( n \), the symbol \([n]\) will denote the set \(\{1, 2, \cdots, n\}\) and for a finite set \(S\), \(|S|\) will denote the number of elements of \(S\).

**Theorem 1.** Let \(G\) be an abelian group of order \(n\) and \(k\) a positive integer. Let \((w_1, w_2, \ldots, w_k)\) be a sequence of integers where each \(w_i\) is co-prime to \(n\). Then, given a sequence \(A: (x_1, x_2, \ldots, x_{k+r})\) of elements of \(G\), where \(1 \leq r \leq n - 1\), if \(0\) is the most repeated element in the sequence, and

\[
\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq 0,
\]

for all permutations \(\sigma\) of \([k+r]\), we have

\[
\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r + 1.
\]

If in the above statement, instead of \(0\), \(x_1\) happens to be the most repeated element, then applying the result on the sequence \((a_1, a_2, \ldots, a_{k+r})\), where \(a_i = x_i - x_1\), for all \(i = 1, 2, \cdots, k+r\), and observing that translation of a subset of \(G\) by an element does not change its cardinality, one obtains the following:

**Corollary 1.** Let \(G\) be an abelian group of order \(n\) and \(k\) a positive integer. Let \((w_1, w_2, \ldots, w_k)\) be a sequence of integers where each \(w_i\) is co-prime to \(n\). Then, given a sequence \(A: (x_1, x_2, \ldots, x_{k+r})\) of elements of \(G\), where \(1 \leq r \leq n - 1\), if \(x_1\) is the most repeated element in the sequence, and

\[
\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \left( \sum_{i=1}^{k} w_i \right) x_1,
\]

for all permutations \(\sigma\) of \([k+r]\), we have

\[
\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r + 1.
\]
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In Corollary 1 above, taking \( r = n - 2 \) (that is, when \( A \) is of length \( k + n - 2 \)), and writing

\[
\alpha = \left( \sum_{i=1}^{k} w_i \right) x_1,
\]

if

\[
\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \alpha,
\]

for all permutations \( \sigma \) of \([k + n - 2]\), we have

\[
\left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k + n - 2] \right\} = G \setminus \{\alpha\}.
\]

Similarly, taking \( r = n - 1 \) in Corollary 1, one obtains the following:

**Corollary 2** (Hamidoune). Let \( G \) be an abelian group of order \( n \) and \( k \) a positive integer. Let \((w_1, w_2, ..., w_k)\) be a sequence of integers where each \( w_i \) is co-prime to \( n \). Then, given a sequence \( A : (x_1, x_2, ..., x_{k+n-1}) \) of elements of \( G \), if \( x_1 \) is the most repeated element in the sequence, we have

\[
\sum_{i=1}^{k} w_i x_{\sigma(i)} = \left( \sum_{i=1}^{k} w_i \right) x_1,
\]

for some permutation \( \sigma \) of \([k + n - 1]\). Hence, if the weights \( w_i \) satisfy

\[
\sum_{i=1}^{k} w_i \equiv 0 \pmod{n}, \quad \text{then we have} \quad \sum_{i=1}^{k} w_i x_{\sigma(i)} = 0,
\]

for some permutation \( \sigma \) of \([k + n - 1]\).

2. Proof of Theorem 1

We need the following result of Scherk [9] (see also [7]).

**Lemma 1.** Let \( B \) and \( C \) be two subsets of an abelian group \( G \) of order \( n \). Suppose \( 0 \in B \cap C \) and suppose that the only solution of

\[
b + c = 0, \quad b \in B, \quad c \in C \quad \text{is} \quad b = c = 0.
\]

Then

\[
|B + C| \geq \min(n, |B| + |C| - 1).
\]
Proof of Theorem 1. Let \( L = \{ i : x_i = 0 \} \) and \( |L| = l \). By our assumption, \( l \leq k - 1 \).

Let \( S \subset [k + r] \setminus L \) be such that \( |S| = s \) is maximal subject to the conditions \( s \leq k - 1 \) and

\[
\sum_{i \in S} w_{f(i)} x_i = 0
\]

for some injective map \( f : S \to [k] \). It is possible that \( S \) is empty.

We note that \( l + s \leq k - 1 \).

For, if \( l + s \geq k \), then \( l \geq k - s \) and hence writing \( S' = \{ j \in [k] : j \neq f(i) \text{ for } i \in S \} \),

\[
\sum_{j \in S'} w_j x_{i_j} = 0 ,
\]

where \( x_{i_j}'s \) run over a subsequence of \( (x_i : i \in L) \) and from (1) and (3), we have

\[
\sum_{i=1}^{k} w_{i} x_{\sigma(i)} = 0 ,
\]

for some permutation \( \sigma \) of \( [k + r] \), contradicting our assumption.

Now, from (2),

\[
|[k + r] \setminus L \cup S| = k + r - (l + s) \geq k + r - (k - 1) = r + 1 .
\]

Therefore, there is \( T \subset [k + r] \setminus L \cup S \) such that \( |T| = r \). Let \( h \) be the maximum number of repetition of any element in the subsequence \( X : (x_i : i \in T) \). By our choice of \( L \), \( h \leq l \) and hence by (2)

\[
h + s \leq l + s \leq k - 1 .
\]

Let \( X = X_1 \cup X_2 \cup \cdots \cup X_h \) be a partition of \( X \) into non-empty subsets, that is, in a particular \( X_i \) no element is repeated. More precisely, this is done in the following way. Let \( x \) be an element of \( X \) which is repeated \( h \) times. Then we put \( x \) in each \( X_i \). Any other element, say \( y \), occurring in \( X \) appears \( m \leq h \) times and we put \( y \) in \( X_i \), \( 1 \leq i \leq m \). Thus,

\[
|X_1| + |X_2| + \cdots + |X_h| = r .
\]

From (4), \( h < k - s \). Let \( w_1', \ldots , w_{k-s}', k \) be the subsequence \( (w_i : i \in S') \), where \( S' = \{ j \in [k] : j \neq f(i) \text{ for } i \in S \} \).

We claim that for \( 1 \leq j \leq h \),

\[
0 \notin w_1'X_1 + w_2'X_2 + \cdots + w_j'X_j .
\]
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If possible, suppose
\[ 0 = w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j}, \]
where \( x_{i_t} \in X_t \), for \( t = 1, 2, \cdots, j \). Then, appending \( w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j} \) to the left hand side of (1), since \( |S \cup \{i_1, i_2, \cdots, i_j\}| = s + j \leq h + s \leq k - 1 \), by (4), we are led to a contradiction to the maximality of \( S \).

This establishes the claim.

Writing \( X'_t = X_t \cup \{0\} \), for \( t = 1, 2, \cdots, h \), and observing that \( |cX'_t| = |X'_t| \), for any integer \( c \) co-prime to \( n \), from (5), we have
\[ \sum_{i=1}^{h} |w'_i X'_i| = r + h. \]

Therefore, by repeated application of Lemma 1 (observe that by (6), the condition of the lemma is satisfied), we have
\[ \left| \sum_{i=1}^{h} w'_i X'_i \right| \geq \min \left\{ n, \sum_{i=1}^{h} |w'_i X'_i| - (h - 1) \right\} \]
\[ = \min \{n, r + 1\} = r + 1. \]

Therefore, \( X: (x_i, i \in T) \) with \( h \) zeros from \( (x_i, i \in L) \) has at least \( (r + 1) \) \( h \)-sums with weights \( w'_i \). So adding a weighted sum of the remaining \( k + r - (r + h) = k - h \) elements of the sequence \( A \) with the remaining \( k - h \) weights to each of the above \( (r + 1) \) \( h \)-sums, we get at least \( (r + 1) k \)-sums with the given weights. \( \square \)

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Sukumar D. Adhikari
Mohan N. Chintamani
Bhavin K. Moriya
Prabal Paul

Harish-Chandra Research Institute
(Former Mehta Research Institute)
Chhatnag Road, Jhusi
Allahabad 211 019
INDIA.
E-mail: adhikari@mri.ernet.in
chintamani@mri.ernet.in
bhavinmoriya@mri.ernet.in
prabal@mri.ernet.in