COMPARISON BETWEEN LOWER AND UPPER $\alpha-$DENSITIES AND LOWER AND UPPER $\alpha-$ANALYTIC DENSITIES

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ABSTRACT. Let $\alpha$ be a real number, with $\alpha \geq -1$. We prove a general inequality between the upper (resp. lower) $\alpha-$analytic density and the upper (resp. lower) $\alpha-$density of a subset $A$ of $\mathbb{N}^*$ (Proposition 2.1). Moreover, we prove by an example that the upper and the lower $\alpha-$densities and the lower and upper $\alpha-$analytic densities of $A$ do not coincide in general (i.e., the inequalities proved in (2.1) may be strict). On the other hand, we identify a class of subsets of $\mathbb{N}^*$ for which these values do coincide in the case $\alpha > -1$.

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1. Introduction

In the whole text $\alpha$ is a real number greater than or equal to $-1$.

Let $A \subseteq \mathbb{N}^*$ be a set of integers. We shall denote by $1_A$ the characteristic function of $A$. Put

$$\mathbb{N}^*_\alpha(n) = \sum_{k \leq n} k^\alpha$$

and

$$D_{A, \alpha}(n) := \frac{\sum_{k \leq n} k^\alpha 1_A(k)}{\mathbb{N}^*_\alpha(n)}; \quad \Delta_{A, \alpha}(t) := t \sum_{k \geq 1} k^\alpha e^{-t\mathbb{N}^*_\alpha(k)} 1_A(k).$$

Define the $\alpha-$density and the $\alpha-$analytic density of $A$, respectively as

$$d_{\alpha}(A) := \lim_{n \to \infty} D_{A, \alpha}(n), \quad \delta_{\alpha}(A) := \lim_{t \to 0^+} \Delta_{A, \alpha}(t),$$


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of course if these limits exist. Notice that \(d_{-1}(A)\) and \(\delta_{-1}(A)\) are, respectively, the classical logarithmic and analytic densities of \(A\) (see [6, pp. 272–273]).

The following theorem (established in [2] in a more general context) links the two concepts of density introduced above (the case \(\alpha = -1\) is well known, see for instance [5, p.274]).

**Theorem 1.3.** Let \(A \subseteq \mathbb{N}^*\) be a set of integers. For every real number \(\ell \in [0,1]\), the two following conditions are equivalent:

(a) \(A\) has \(\alpha\)–density \(d_\alpha(A) = \ell\).

(b) \(A\) has \(\alpha\)–analytic density \(\delta_\alpha(A) = \ell\).

When the \(\alpha\)–density (resp. the \(\alpha\)–analytic density) of \(A\) doesn’t exist, one can consider the upper and lower densities, defined as follows.

**Definition 1.4.** The lower and upper \(\alpha\)–densities of \(A\) are defined respectively as

\[
d_{\alpha}(A) := \lim \inf_{n \to \infty} D_{A,\alpha}(n), \quad \overline{d}_{\alpha}(A) := \lim \sup_{n \to \infty} D_{A,\alpha}(n).
\]

When \(d_{\alpha}(A) = \overline{d}_{\alpha}(A)\) (i.e., when the limit \(d_{\alpha}(A) := \lim_{n \to \infty} D_{A,\alpha}(n)\) exists), we say that \(A\) has \(\alpha\)–density equal to \(d_{\alpha}(A)\).

**Definition 1.5.** The lower and upper \(\alpha\)–analytic densities of \(A\) are defined respectively as

\[
\underline{\delta}_{\alpha}(A) := \lim \inf_{t \to 0^+} \Delta_{A,\alpha}(t), \quad \overline{\delta}_{\alpha}(A) := \lim \sup_{t \to 0^+} \Delta_{A,\alpha}(t).
\]

When \(\underline{\delta}_{\alpha}(A) = \overline{\delta}_{\alpha}(A)\) (i.e., when the limit \(\delta_{\alpha}(A) := \lim_{t \to 0^+} \Delta_{A,\alpha}(t)\) exists), we say that \(A\) has \(\alpha\)–analytic density equal to \(\delta_{\alpha}(A)\).

In this paper we outface the problem of the relation between lower and upper \(\alpha\)–densities and lower and upper \(\alpha\)–analytic densities. In Section 2 we prove that, in general, the upper \(\alpha\)–analytic density of \(A\) is not greater than the upper \(\alpha\)–density of \(A\) (concerning the lower densities, the inequality is obviously reversed). This is done in Proposition 2.1.

A natural question is whether the same kind of result as Theorem 1.3 can be stated also for lower and upper \(\alpha\)–densities and lower and upper \(\alpha\)–analytic densities, i.e., whether the upper and the lower \(\alpha\)–densities and the lower and upper \(\alpha\)–analytic densities of \(A\) coincide. In Section 3 of this paper we prove
that the answer is negative in general: see Theorem 3.1. On the other hand, in Section 4 we identify a class of subsets of N* for which the question can be answered affirmatively if \( \alpha > -1 \).

The case \( \alpha = -1 \) is open.

2. A first result

In this section we prove the following proposition.

**Proposition 2.1.** For every \( A \subseteq \mathbb{N}^* \), the following inequalities hold:

\[
d_\alpha(A) \leq \hat{\delta}_\alpha(A) \leq \tilde{\delta}_\alpha(A) \leq \tilde{d}_\alpha(A).
\]

**Proof.** Let \( A \subseteq \mathbb{N}^* \) and \( \alpha \geq -1 \) be fixed. For \( x \geq 1 \), put

\[
A_\alpha(x) = \sum_{k \leq x} k^\alpha 1_A(k), \quad S_\alpha(x) = \begin{cases} 
\frac{x^{\alpha+1}}{\alpha+1} & \text{for } \alpha > -1, \\
\log x & \text{for } \alpha = -1.
\end{cases}
\]

At first we shall prove the following lemma.

**Lemma 2.2.** The following relation holds:

\[
\sum_{k \in \mathbb{N}^*} k^\alpha 1_A(k)e^{-tS_\alpha(k)} = t \int_1^\infty A_\alpha(x) x^\alpha e^{-tS_\alpha(x)}dx.
\]

**Proof.** We simplify the notations and write \( A(x) \) and \( S(x) \) instead of \( A_\alpha(x) \) and \( S_\alpha(x) \) respectively. For every integer \( m > 1 \) we have

\[
t \int_1^m A(x) x^\alpha e^{-tS(x)}dx = t \sum_{k=1}^{m-1} \int_k^{k+1} A(x) x^\alpha e^{-tS(x)}dx
\]

\[
= \sum_{k=1}^{m-1} A(k) \int_k^{k+1} tx^\alpha e^{-tS(x)}dx.
\]
By performing the change of variable \( y = tS(x) \) in the integral, the above quantity is transformed into
\[
\sum_{k=1}^{m-1} A(k) \int_{tS(k)}^{tS(k+1)} e^{-y} dy = \sum_{k=1}^{m-1} A(k) \left[ e^{-tS(k)} - e^{-tS(k+1)} \right]
\]
\[
= A(1)e^{-tS(1)} + \sum_{k=2}^{m-1} A(k)e^{-tS(k)} - \sum_{k=1}^{m-2} A(k)e^{-tS(k+1)} - A(m-1)e^{-tS(m)}
\]
\[
= A(1)e^{-tS(1)} + \sum_{k=2}^{m-1} (A(k) - A(k-1))e^{-tS(k)} - A(m-1)e^{-tS(m)}
\]
\[
= \sum_{k=1}^{m-1} k^\alpha 1_A(k)e^{-tS(k)} - A(m-1)e^{-tS(m)},
\]
since \( A(1) = 1^\alpha 1_A(1) \) and, for \( k \geq 2, A(k) - A(k-1) = k^\alpha 1_A(k) \). We obtain the statement by passing to the limit as \( m \to \infty \), since
\[
0 \leq A(m - 1) \leq N^*_\alpha(m) \sim S(m), \quad m \to \infty.
\]  
\[\Box\]

**Proof of Proposition 2.1 continued.** We use the (simplified) notations of Lemma 2.2 and prove only the last inequality: the first one follows since \( \underline{\delta}_\alpha(A) = 1 - \overline{\delta}_\alpha(\mathbb{N}^*_\alpha \setminus A) \) and \( \overline{\delta}_\alpha(A) = 1 - \underline{\delta}_\alpha(\mathbb{N}^*_\alpha \setminus A) \); the second one is obvious. For every \( \epsilon > 0 \), there exists an integer \( \nu \) such that, for every \( n \geq \nu \), we have
\[
\frac{A(n)}{S(n)} \leq \overline{\delta}_\alpha(A) + \epsilon.
\]  
(2.4)

Hence, by Lemma 2.2, we have
\[
t \sum_k k^\alpha 1_A(k)e^{-tS(k)} = t^2 \int_1^\infty A(x) x^\alpha e^{-tS(x)} dx = t^2 \int_1^\infty A(\lfloor x \rfloor) x^\alpha e^{-tS(x)} dx
\]
\[
\leq t^2 \left( \int_1^\nu A(\lfloor x \rfloor) x^\alpha e^{-tS(x)} dx + (\overline{\delta}_\alpha(A) + \epsilon) \int_\nu^\infty S(\lfloor x \rfloor) x^\alpha e^{-tS(x)} dx \right)
\]
\[
\leq t^2 \left( \int_1^\nu A(\lfloor x \rfloor) x^\alpha e^{-tS(x)} dx + (\overline{\delta}_\alpha(A) + \epsilon) \int_\nu^\infty S(x) x^\alpha e^{-tS(x)} dx \right).
\]  
(2.5)

Now
\[
0 \leq t^2 \int_1^\nu A(\lfloor x \rfloor) x^\alpha e^{-tS(x)} dx \leq t^2 \int_1^\nu A(\lfloor x \rfloor) x^\alpha dx \leq t^2 A(\nu) \nu^\alpha (\nu - 1),
\]  
(2.6)
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while

$$t^2 \int_0^\infty S(x) x^\alpha e^{-tS(x)} \, dx = \int_0^\infty (tS(x)) (tx^\alpha) e^{-tS(x)} \, dx$$

$$\leq \int_0^\infty (tS(x)) (tx^\alpha) e^{-tS(x)} \, dx = \int_0^\infty ye^{-y} \, dy = 1.$$ (2.7)

By using (2.6) and (2.7) and passing to the limsup as $t \to 0$ in (2.5) we obtain

$$\bar{d}_\alpha(A) = \limsup_{t \to 0} t \sum_{k=1}^n k^\alpha 1_A(k) e^{-tS(k)} \leq \bar{d}_\alpha(A) + \epsilon,$$

and the statement follows by the arbitrariness of $\epsilon$. \hfill \Box

3. A second result

In general the upper and the lower $\alpha$–densities and the lower and upper $\alpha$–analytic densities of $A$ do not coincide (i.e., the inequalities of Proposition 2.1 may be strict). We prove it hereafter for $\alpha = 0$.

**Theorem 3.1.** There is a subset $A$ of $\mathbb{N}^*$ such that $\bar{d}_0(A) < \bar{d}_0(A)$.

**Proof.** Let $r$ be a real number, with $0 < r \leq 1$, and put $p_n = 10^n$, $q_n = \lfloor 10^n + r \rfloor$. We consider the set

$$A = \bigcup_{n \geq 1} ([p_n, q_n] \cap \mathbb{N}^*).$$

It is not difficult to calculate $\bar{d}_0(A)$ and $\bar{d}_0(A)$ using the following proposition, proved in [2] in a more general context.

**Proposition 3.2.** Let $A$ be a subset of $\mathbb{N}^*$, given in the form

$$A = \bigcup_{n \geq 1} ([p_n, q_n] \cap \mathbb{N}^*).$$

Put

$$\rho_k = q_k - p_k, \quad \sigma_k = q_k - q_{k-1}, \quad k \geq 1 \quad (q_0 = 1).$$

Then

$$\bar{d}_0(A) = \limsup_{n \to \infty} \frac{\sum_{k=1}^n \rho_k}{\sum_{k=1}^n \sigma_k}, \quad \underline{d}_0(A) = \liminf_{n \to \infty} \frac{q_{n-1} - \sum_{k=1}^{n-1} \rho_k}{p_n - \sum_{k=1}^{n-1} \sigma_k}.$$
For the set $A$ in (3.1), from the relations
\[ x \leq \lfloor x \rfloor \leq x + 1 \] (3.3)
we find easily
\[ \sum_{k=1}^{n} \sigma_k = q_n - 1 = \lfloor 10^{n+r} \rfloor - 1 \sim 10^{n+r}. \] (3.4)

On the other hand
\[ \sum_{k=1}^{n} \rho_k = \sum_{k=1}^{n} (\lfloor 10^{k+r} \rfloor - 10^k) \]
and, using (3.3) again, we have
\[ \sum_{k=1}^{n} (10^{k+r} - 10^k) - n \leq \sum_{k=1}^{n} \rho_k \leq \sum_{k=1}^{n} (10^{k+r} - 10^k). \] (3.5)

Since
\[ \sum_{k=1}^{n} (10^{k+r} - 10^k) = (10^r-1) \sum_{k=1}^{n} 10^k = \frac{10}{9} (10^r-1)(10^n-1), \]
from (3.5) we get
\[ \sum_{k=1}^{n} \rho_k \sim \frac{10}{9} (10^r-1)(10^n-1), \] (3.6)

and finally, from (3.4) and (3.6), we conclude that
\[ d_0(A) = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \rho_k}{\sum_{k=1}^{n} \sigma_k} = \frac{10}{9} (1 - 10^{-r}). \] (3.7)

**Remark 3.8.** With the same technique we can calculate also $d_0(A)$, obtaining
\[ d_0(A) = \frac{10^r - 1}{9}. \]

We now show that, for sufficiently small values of $r$, the lower and upper 0–analytic densities of $A$ are different from the above calculated values. We have
\[ t \sum_{k \in \mathbb{N}^*} e^{-tk} 1_A(k) = t \sum_{n \geq 1} \left( \sum_{k=p_n}^{q_n-1} e^{-tk} \right) = \frac{t}{1 - e^{-t}} \sum_{n \geq 1} (e^{-tp_n} - e^{-tq_n}). \] (3.9)

Since $t/(1 - e^{-t}) \to 1$ as $t \to 0$, the two functions
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\[
t \mapsto t \sum_{k \in \mathbb{N}^*} e^{-tk} 1_A(k); \quad t \mapsto \sum_{n \geq 1} (e^{-tp_n} - e^{-tq_n})
\]

have the same limit points as $t \to 0^+$.

It follows that the values of $\delta_0(A)$ and $\hat{\delta}_0(A)$ are given respectively by

\[
\limsup_{t \to 0^+} \sum_{n \geq 1} (e^{-tp_n} - e^{-tq_n}); \quad \liminf_{t \to 0^+} \sum_{n \geq 1} (e^{-tp_n} - e^{-tq_n}),
\]

and in the sequel we shall estimate these two quantities. Obviously

\[
\sum_{k \in \mathbb{N}^*} (e^{-tk} - e^{-t(k+1)}) \leq \sum_{k \in \mathbb{N}^*} (e^{-t10^k} - e^{-t10^{k+r}}),
\]  

and we are going to evaluate each term in the second sum above. By Lagrange’s Theorem, for every integer $k$ there exists a real number $x_k$, with $k \leq x_k \leq k + r$, such that

\[
\left( e^{-t10^k} - e^{-t10^{k+r}} \right) = (-r) \frac{d}{dx} \left( e^{-t10^x} \right) \bigg|_{x=x_k} = (\log 10) rt 10^{x_k} e^{-t10^{x_k}}
\]

\[
\leq (\log 10) rt 10^{k+r} e^{-t10^k} = (\log 10) rt 10^{r} (10^k e^{-t10^k}).
\]  

From (3.10) and (3.11) we get

\[
\sum_{k \in \mathbb{N}^*} \left( e^{-tk} - e^{-t(k+1)} \right) 1_A(k) \leq (\log 10) rt 10^r \sum_{k \in \mathbb{N}^*} 10^k e^{-t10^k},
\]

and now we want to find an upper bound for the last series. We can confine ourselves to the case $t \in (0, 1)$. Put $x_t = -\log_{10} t$; then it is easy to see that the function

\[
x \mapsto 10^r e^{-t10^x}
\]

has a unique absolute maximum in $x = x_t = -\log_{10} t$, which means that it is strictly increasing for $x \in (0, x_t]$ and strictly decreasing for $x \in [x_t, +\infty)$. Then, applying twice the theorem of comparison of a sum and an integral for monotone functions (see Theorem 2, p. 4 of [6]) we get, for every integer $n > x_t$ and for two suitable numbers $\theta_1$ and $\theta_2$ in $[0, 1]$,
\[
\sum_{0 < k \leq n} 10^k e^{-t10^k} = \sum_{0 < k \leq \lceil x \rceil} 10^k e^{-t10^k} + \sum_{\lceil x \rceil < k \leq n} 10^k e^{-t10^k}
\]
\[
= \int_0^{\lceil x \rceil} 10^x e^{-t10^x} \, dx + \theta_1 (10^\lceil x \rceil e^{-t10^\lceil x \rceil} - e^{-t})
\]
\[
+ \int_{\lceil x \rceil}^n 10^x e^{-t10^x} \, dx + \theta_2 (10^n e^{-t10^n} - 10^\lceil x \rceil e^{-t10^\lceil x \rceil})
\]
\[
= \int_0^n 10^x e^{-t10^x} \, dx + \theta_1 (10^\lceil x \rceil e^{-t10^\lceil x \rceil} - e^{-t})
\]
\[
+ \theta_2 (10^n e^{-t10^n} - 10^\lceil x \rceil e^{-t10^\lceil x \rceil})
\]
\[
\leq \int_0^n 10^x e^{-t10^x} \, dx + 10^\lceil x \rceil e^{-t10^\lceil x \rceil} + 10^n e^{-t10^n}
\]
\[
= \int_0^n 10^x e^{-t10^x} \, dx + \frac{1}{et} + 10^n e^{-t10^n}.
\]

By passing to the limit with respect to \( n \), we conclude that
\[
\sum_{k \in \mathbb{N}^*} 10^k e^{-t10^k} \leq \int_0^\infty 10^x e^{-t10^x} \, dx + \frac{1}{et} = \frac{1}{\log 10} \int_1^\infty e^{-ty} \, dy + \frac{1}{et}.
\]

Going back to relation (3.12), and using (3.13), we find that
\[
\sum_{k \in \mathbb{N}^*} (e^{-tk} - e^{-t(k+1)}) 1_A(k) \leq r 10^r \left(1 + \frac{\log 10}{e}\right),
\]
hence also
\[
\bar{\delta}_0(A) = \limsup_{t \to 0^+} \sum_{k \in \mathbb{N}^*} (e^{-tk} - e^{-t(k+1)}) 1_A(k) \leq r 10^r \left(1 + \frac{\log 10}{e}\right).
\]

Now we prove that, for sufficiently small \( r \), we have \( \bar{\delta}_0(A) \neq \underline{\delta}_0(A) \). Recalling relations (3.14) and (3.7), it will be enough to prove that
\[
r 10^r \left(1 + \frac{\log 10}{e}\right) < \frac{10}{9} (1 - 10^{-r})
\]
for sufficiently small \( r \), and this is clearly true since
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\[
\lim_{r \to 0} \frac{10(1 - 10^{-r})}{9r10^r} = \frac{10}{9} \log 10 \approx 2.55, \quad \text{while} \quad \left(1 + \frac{\log 10}{e}\right) \approx 1.847.
\]

\(\square\)

**Remark 3.15.** For the lower densities of \(A\), the same technique gives

\[\delta_0(A) \geq r10^{-r} > \frac{10r - 1}{9} = d_0(A),\]

for sufficiently small \(r\).

4. The case of regular sets

In this Section we shall exhibit a class of subsets of \(\mathbb{N}^*\) for which the values of the lower and upper \(\alpha\)-densities and of the lower and upper \(\alpha\)-analytic densities coincide. Recall that the counting function of \(A\) is the function defined as

\[A(x) := \text{card}\{k \in A : k \leq x\} = \sum_{k \leq x} 1_A(k), \quad x \in \mathbb{R}.
\]

Obviously

\[1_A(n) = A(n) - A(n - 1), \quad n \in \mathbb{N}^*.
\]

**Definition 4.1.** A subset \(A\) of \(\mathbb{N}^*\) is regular if its counting function \(A(x)\) has the form

\[A(x) = L(x)x^\gamma,
\]

for a suitable positive slowly varying function \(L\) and a suitable real number \(\gamma \in [0, 1]\). A function \(L : [0, +\infty] \rightarrow [0, +\infty]\) is called slowly varying if for any \(t > 0\), \(\frac{L(tx)}{L(x)}\) tends to 1, as \(x\) tends to +\(\infty\). \(\gamma\) is called the regularity exponent of \(A\) (see [5]).

If \(\gamma < 1\), then obviously \(A\) has 0-density and \(d_0(A) = 0\). Moreover, by Cor. 6.9 and Ex. 6.10 p. 271 of [2], \(A\) has \(\alpha\)-density for every \(\alpha > -1\) and \(d_\alpha(A) = 0\).

If \(\gamma = 1\) things go differently. In fact, in this case the regularity assumption doesn’t imply that \(A\) has 0-density, as it is shown by an example constructed in [3] (Appendix, p. 190). By the above quoted results (i.e., Cor. 6.9 and Ex. 6.10, p. 271 of [2]), such a set \(A\) cannot have \(\alpha\)-density for any \(\alpha > -1\) either. These remarks motivates our main result, here below.

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Let $\alpha > -1$; let $A$ be a regular subset of $\mathbb{N}^*$, with regularity exponent $\gamma = 1$. For every real number $\ell \in [0, 1]$, the following two conditions are equivalent:

(4.2)(a) $\ell$ is a cluster point for the sequence
$$n \mapsto D_{A,\alpha}(n), \quad \text{as } n \to \infty;$$

(4.2)(b) $\ell$ is a cluster point for the function
$$t \mapsto \Delta_{A,\alpha}(t), \quad \text{as } t \to 0^+.$$

**Theorem 4.2.** Let $\alpha > -1$; let $A$ be a regular subset of $\mathbb{N}^*$, with regularity exponent $\gamma = 1$. Then
$$\delta_0(A) = d_0(A) \quad \text{and} \quad \delta_0(A) = d_0(A).$$

In order to give the proof of Theorem 4.2, we need some preliminaries. We start by recalling a well known tauberian theorem (proved in [1, pp. 443-444]).

**Theorem 4.4.** Let $F$ be a distribution function with Laplace transform $\phi$, i.e.,
$$\phi(t) = \int_0^\infty e^{-tx} F(dx).$$

We assume that $\phi(t)$ is finite for every $t$ in a right neighborhood of 0. Let $\rho$ be a real number, with $\rho \geq 0$. The two following conditions are equivalent:

$$\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^\rho; \quad \text{(4.4)(a)}$$
$$\lim_{t \to 0^+} \frac{\phi(t\lambda)}{\phi(t)} = \frac{1}{\lambda^\rho}. \quad \text{(4.4)(b)}$$
Moreover, each of them implies that
\[ \phi(t) \sim \Gamma(\rho + 1)F(t^{-1}), \quad \text{as} \quad t \to 0^+. \tag{4.5} \]

**Remark 4.6.** Condition (4.4)(a) (resp. (4.4)(b)) means that \( F \) (resp \( \phi \)) is regularly varying with exponent \( \rho \) (resp. \(-\rho\)), as \( x \to \infty \) (resp. as \( t \to 0^+ \)).

An immediate consequence of Theorem 4.4 is the following.

**Proposition 4.7.** Assume that either of the equivalent conditions of Theorem 4.4 holds, and let \( \ell \geq 0 \). The two following conditions are equivalent:

- (4.7)(a) \( \ell \) is a cluster point for the function \( x \mapsto \frac{F(x)}{x} \), as \( x \to \infty \);
- (4.7)(b) \( \Gamma(\rho + 1) \cdot \ell \) is a cluster point for the function \( t \mapsto t\phi(t) \), as \( t \to 0^+ \).

**Proof of Proposition 4.7.** (a)\( \Rightarrow \) (b). Let \( (x_n)_{n \geq 1} \) be a sequence of real numbers such that
\[ \lim_{n \to \infty} x_n = \infty; \quad \lim_{n \to \infty} \frac{F(x_n)}{x_n} = \ell. \]

Put \( t_n = x_n^{-1} \). Then \( \lim_{n \to \infty} t_n = 0 \) and from relation (4.5) we deduce
\[ \lim_{n \to \infty} t_n\phi(t_n) = \lim_{n \to \infty} t_n F(t_n^{-1})\Gamma(\rho + 1) = \lim_{n \to \infty} \frac{F(x_n)}{x_n} \Gamma(\rho + 1) = \Gamma(\rho + 1) \cdot \ell. \]

(b)\( \Rightarrow \) (a) is similar. □

**Proof of Theorem 4.4.** Now we are ready to prove Theorem 4.4. Consider the measure on \( \mathbb{N}^* \) defined as
\[ \mu = \sum_{k \in \mathbb{N}^*} k^\alpha 1_A(k)\epsilon_{N^*_k(k)}, \]
where \( \epsilon_x \) denoted the Dirac mass placed in the point \( x \). The distribution function of \( \mu \) is
\[ F(x) = \sum_{k : N^*_k(k) \leq x} k^\alpha 1_A(k), \tag{4.8} \]
while its Laplace transform is given by the formula
\[ \phi(t) = \sum_{k \in \mathbb{N}^*} k^\alpha 1_A(k)e^{-tN^*_k(k)}. \]

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The abscissa of convergence $\sigma_0$ of the Dirichlet series defining $\phi(t)$ is given by the well known formula (see [4])

$$\sigma_0 = \limsup_{n \to \infty} \frac{\log \left( \sum_{k=1}^{n} k^\alpha 1_A(k) \right)}{N_\alpha^*(n)},$$

which equals 0 since

$$0 \leq \frac{\log \left( \sum_{k=1}^{n} k^\alpha 1_A(k) \right)}{N_\alpha^*(n)} \leq \frac{\log \left( \sum_{k=1}^{n} k^\alpha \right)}{N_\alpha^*(n)} = \frac{\log N_\alpha^*(n)}{N_\alpha^*(n)}.$$

Hence $\phi$ is defined for every $t > 0$ and the first assumption of Theorem 4.4 is satisfied.

By Proposition 4.7, the Theorem will be proved if we show that

(i) $F$ satisfies assumption (4.4) (a) with exponent $\rho = 1$;

(ii) assumption (4.2) (a) is equivalent to assumption (4.7)(a).

(i) By Abel’s summation formula and (4.1) we have, for every integer $n$, 

$$\sum_{k \leq n} k^\alpha 1_A(k) = A(n)n^\alpha - \sum_{k \leq n-1} A(k)((k+1)^\alpha - k^\alpha)$$

$$= n^{\alpha+1}L(n)(1 - M(n))$$

where

$$M(n) = \frac{\sum_{k \leq n-1} kL(k)((k+1)^\alpha - k^\alpha)}{n^{\alpha+1}L(n)}.$$ 

We prove that

$$\lim_{n \to \infty} M(n) = \frac{\alpha}{\alpha + 1}.$$ 

Put

$$R(n) = \frac{(n+1)^\alpha - n^\alpha}{\alpha n^{\alpha-1}}.$$ 

Then, by an application of Lagrange’s Theorem it is easily seen that

$$\lim_{n \to \infty} R(n) = 1,$$

so that, for every $\epsilon > 0$, there exists an integer $n_0$ such that, for every integer $n > n_0$, we have $1 - \epsilon < R(n) < 1 + \epsilon$. Take $n > n_0 + 1$. Then we can write

$$M(n) = \alpha \frac{\sum_{k \leq n-1} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)}$$

$$= \alpha \frac{\sum_{k \leq n_0} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)} + \alpha \frac{\sum_{n_0 < k \leq n-1} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)}.$$
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The first fraction above goes to 0 as $n \to \infty$, since $\lim_{n \to \infty} n^{\alpha+1} L(n) = \infty$. As to the second one, we have

\[
(1 - \epsilon) \frac{\sum_{n_0 < k \leq n-1} k^\alpha L(k)}{n^{\alpha+1} L(n)} \leq \frac{\sum_{n_0 < k \leq n-1} k^\alpha L(k) R(k)}{n^{\alpha+1} L(n)} \leq (1 + \epsilon) \frac{\sum_{n_0 < k \leq n-1} k^\alpha L(k)}{n^{\alpha+1} L(n)}.
\]

By Lemma 4.8, p. 180 of [3], the first and last fractions above go to $(\alpha + 1)^{-1}$ as $n \to \infty$, and we conclude by the arbitrariness of $\epsilon$.

Now put

\[
n_x = \max \{ k \in \mathbb{N}^*: N_\alpha^*(k) \leq x \},
\]

so that we can write

\[
F(x) = \sum_{k: N_\alpha^*(k) \leq x} k^\alpha 1_A(k) = \sum_{k \leq n_x} k^\alpha 1_A(k).
\] (4.11)

The two relations

\[N_\alpha^*(n) \sim N_\alpha^*(n + 1), \quad n \to \infty \quad \text{and} \quad N_\alpha^*(n_x) \leq x < N_\alpha^*(n_x + 1)\]

yield that

\[N_\alpha^*(n_x) \sim x, \quad x \to \infty.\] (4.12)

On the other hand, from (4.12) and the relation

\[N_\alpha^*(n) \sim \frac{n^{\alpha+1}}{\alpha + 1},\]

we deduce that

\[
\frac{n_x^{\alpha+1}}{\alpha + 1} \sim x,
\]
or, equivalently

\[n_x \sim ((\alpha + 1)x)^{(\alpha+1)^{-1}}.\] (4.13)

Recalling (4.11), we can use (4.13) in (4.9) and, from (4.10) and the fact that $L$ is slowly varying we get immediately

\[
\lim_{t \to \infty} \frac{F(xt)}{F(x)} = x,
\]
i.e., the claim of point (i).

(ii) (4.2)(a) $\Rightarrow$ (4.7)(a). Assume that $\ell$ is a limit point for the sequence

\[
n \mapsto \frac{\sum_{k \leq n} k^\alpha 1_A(k)}{N_\alpha^*(n)}, \quad \text{as} \quad n \to \infty;
\]
since (see (4.8))
\[ \sum_{k \leq n} k^\alpha 1_A(k) = \sum_{N_\alpha^*(k) \leq N_\alpha^*(n)} k^\alpha 1_A(k) = F(N_\alpha^*(n)), \]
the above statement is clearly equivalent to the statement that \( \ell \) is a limit point for the sequence
\[ n \mapsto \frac{F(N_\alpha^*(n))}{N_\alpha^*(n)}, \quad \text{as} \quad n \to \infty, \]
which in turn implies (4.7)(a).

(4.7)(a) \( \Rightarrow \) (4.2)(a). Assume that \( \ell \) is a limit point for the function
\[ x \mapsto \frac{F(x)}{x}, \quad \text{as} \quad x \to \infty, \]
and let \((x_k)\) be a sequence of numbers such that \( \lim_{k \to \infty} x_k = +\infty \) and
\[ \lim_{k \to \infty} \frac{F(x_k)}{x_k} = \ell. \]
In order to simplify the notations, we put \( N_\alpha^*(n) = S(n) \). Denote by the same symbol \( S \) the function defined on \( \mathbb{R}^+ \) as
\[ S(x) = \begin{cases} S(n) & \text{for } x = n, \forall n \in \mathbb{N}^*; \\ \text{linear} & \text{elsewhere}. \end{cases} \]
Then \( S \) is strictly increasing, hence invertible; let \( S^{-1} \) its inverse and put \( n_k = \lfloor S^{-1}(x_k) \rfloor \). We want to prove that
\[ \lim_{k \to \infty} \frac{F(S(n_k))}{S(n_k)} = \ell. \] (4.14)
We can write
\[ \frac{F(S(n_k))}{S(n_k)} = \frac{F(S(n_k))}{S(n_k + 1)} \cdot \frac{S(n_k + 1)}{S(n_k)}. \] (4.15)
The second fraction above goes to 1 as \( k \to \infty \) since
\[ S(n) \sim \frac{n^\alpha}{\alpha + 1} \sim \frac{(n + 1)^\alpha}{\alpha + 1} \sim S(n + 1), \quad n \to \infty. \]
As to the first one, we observe that, since \( n_k \leq S^{-1}(x_k) < n_k + 1 \), we have
\[ S(n_k) \leq x_k \leq S(n_k + 1) \]
hence, since \( F \) is non-decreasing,
\[ \limsup_{k \to \infty} \frac{F(S(n_k))}{S(n_k + 1)} \leq \limsup_{k \to \infty} \frac{F(x_k)}{x_k} = \ell. \] (4.16)
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From (4.15) and (4.16) we conclude that

\[
\limsup_{k \to \infty} \frac{F(S(n_k))}{S(n_k)} \leq \ell. \tag{4.17}
\]

A similar argument shows that

\[
\liminf_{k \to \infty} \frac{F(S(n_k))}{S(n_k)} \geq \ell, \tag{4.18}
\]

hence, from (4.17) and (4.18)

\[
\lim_{k \to \infty} \frac{\sum_{k < n_k} k^\alpha 1_A(k)}{S(n_k)} = \lim_{k \to \infty} \frac{F(S(n_k))}{S(n_k)} = \ell,
\]

which concludes the proof of (ii) and of Theorem 4.2. \( \square \)

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