ALGEBRAIC NUMBERS AND DENSITY
MODULO 1, II

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ABSTRACT. This is a companion paper to [8]. In [8], using ideas of Berend [3] and Kra [6], it was proved that the sets of the form
\[ \{ \lambda_1^m \mu_1^m \xi_1 + \lambda_2^m \mu_2^m \xi_2 : n, m \geq 1 \}, \]
where \( \xi_1, \xi_2 \in \mathbb{R}, \lambda_1, \mu_1 \) and \( \lambda_2, \mu_2 \) are two pairs of multiplicatively independent real algebraic numbers satisfying certain technical conditions, including that \( \mu_i \in \mathbb{Q}(\lambda_i), i = 1, 2, \) are dense modulo \( 1/\kappa, \) for some \( \kappa \geq 1. \)

In this paper we extend the result from [8], showing that the condition \( \mu_i \in \mathbb{Q}(\lambda_i) \) can be removed by imposing appropriate conditions on the norms of conjugates of \( \lambda_i, \mu_i \) and the degree of the algebraic numbers \( \lambda_i^n \mu_i^m. \)

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1. Introduction

This is a companion paper to [8]. We extend the result from [8] showing that the condition \( \mu_i \in \mathbb{Q}(\lambda_i) \) can be removed by imposing appropriate conditions on the norms of conjugates of \( \lambda_i, \mu_i \) and the degree of the algebraic numbers \( \lambda_i^n \mu_i^m. \) In order to do that we define in §3.1 the semigroup \( \Sigma \) in a different way than it was done in [8]. Having this new semigroup \( \Sigma \) the main steps of the proof of our result remain essentially the same as in [8]. However, some modifications are required. For example, we have to deal with the higher dimensional dynamical


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systems than that of $\mathbb{S}$, and hence the proof becomes much more technical and complicated.

Let $K$ be a real algebraic number field, and let $K^*$ denote its multiplicative group. Recall that two numbers $\lambda, \mu \in K^*$ are called multiplicative dependant if there exist integers $m$ and $n$, not both of which are 0 with $\lambda^m = \mu^n$. Equivalently, they are both rational powers of the same element $\beta \in K^*$. We say that $\lambda$ and $\mu$ are multiplicatively independent if they are not multiplicatively dependent.

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

**Theorem 1.1.** Let $\lambda_1, \mu_1$ and $\lambda_2, \mu_2$ be two distinct pairs of multiplicatively independent real algebraic numbers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that

(i) for every $n, m \in \mathbb{N}$, $\deg_Q(\lambda_i^n \mu_i^m) = 4$.

Let $p_1, p_2, \ldots, p_s \geq 2$ be the primes appearing in the denominators of coefficients of the minimal polynomials $P_{\lambda_i}, P_{\mu_i}, i \in \mathbb{Q}[x]$ of $\lambda_i$ and $\mu_i$, $i = 1, 2$. We set $S = \{\infty, p_1, p_2, \ldots, p_s\}$.

Assume further that the following conditions are satisfied:

(ii) $|\lambda_i|_\infty > |\tilde{\lambda}_i|_\infty > 1$ and $|\mu_i|_\infty > |\tilde{\mu}_i|_\infty > 1$, $i = 1, 2$,

(iii) there exist $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, and two positive integers $k, l$, $k \neq l$ such that

\[
\min \left( \min_{p \in S \setminus \{\infty\}} |\lambda_2 \mu_2|_{p}^{\alpha} |\lambda_2 \mu_2|_{p}^{k} |\tilde{\lambda}_2 \tilde{\mu}_2|_{\infty}^{\alpha} |\tilde{\lambda}_2 \tilde{\mu}_2|_{\infty}^{k} \right) > \max \left( \max_{p \in S \setminus \{\infty\}} |\lambda_1 \mu_1|_{p}^{\alpha} |\lambda_1 \mu_1|_{p}^{l} |\tilde{\lambda}_1 \tilde{\mu}_1|_{\infty}^{\alpha} |\tilde{\lambda}_1 \tilde{\mu}_1|_{\infty}^{l} \right)
\]

and

\[
\min \left( \min_{p \in S \setminus \{\infty\}} |\lambda_1 \mu_1|_{p}^{\alpha'} |\lambda_1 \mu_1|_{p}^{k} |\tilde{\lambda}_1 \tilde{\mu}_1|_{\infty}^{\alpha'} |\tilde{\lambda}_1 \tilde{\mu}_1|_{\infty}^{k} \right) > \max \left( \max_{p \in S \setminus \{\infty\}} |\lambda_2 \mu_2|_{p}^{\alpha'} |\lambda_2 \mu_2|_{p}^{l} |\tilde{\lambda}_2 \tilde{\mu}_2|_{\infty}^{\alpha'} |\tilde{\lambda}_2 \tilde{\mu}_2|_{\infty}^{l} \right),
\]

(iv) $|\lambda_i|_p = |\tilde{\lambda}_i|_p > 1$ and $|\mu_i|_p = |\tilde{\mu}_i|_p > 1$, for $p \in S \setminus \{\infty\}$.

Then for any pair of real numbers $\xi_1, \xi_2$, with at least one $\xi_i$ non-zero, there exists a natural number $\kappa$ such that the set

\[
\{ \lambda_1^n \mu_1^m \kappa \xi_1 + \lambda_2^n \mu_2^m \kappa \xi_2 : n, m \in \mathbb{N} \}
\]

is dense modulo 1.

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1Here $| \cdot |_p$ stands for the $p$-adic norm in the field $\mathbb{Q}_p(\lambda_1, \mu_1, \lambda_2, \mu_2)$, whereas $| \cdot |_\infty$ denotes the usual absolute value in $\mathbb{R}$. 

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As a result we are able to consider more general expressions containing algebraic numbers than that considered in [8]. For example, our Theorem 1.1 implies that the following double-sequence
\[
\left(7\sqrt{2} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 7}\right)^n \left(\frac{7^2}{\sqrt{5}} + \frac{1}{2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2}\right)^m + \\
\left(7^5\sqrt{3} + \frac{1}{2^{11} \cdot 3^{11} \cdot 5^{11} \cdot 7^{11}}\right)^n \left(\frac{7}{\sqrt{p}} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 7}\right)^m, \quad n, m \in \mathbb{N}
\]
(1.3)
is dense modulo $1/\kappa$ for some $\kappa \geq 1$ (see Corollary 2.9).

**Remark.** We should remark here that Theorem 1.1 is not a generalization of [8, Theorem 1.5]. These two theorems are of different kind. In particular, it is not true that if the algebraic numbers $\lambda_i, \mu_i$, $i = 1, 2$, satisfy the set of conditions required in [8] then they satisfy all of the assumptions required here. For instance, in the example on p. 647 in [8], illustrating [8, Theorem 1.5], one has $\text{deg}_Q \lambda_i \mu_i = 2$, $i = 1, 2$. Hence, the condition (i) of Theorem 1.1 is not satisfied.

**Remark.** Although conditions given in (iii) seem to be very complicated they have in fact a simple dynamical meaning. Namely, the various norms may be understood as the speeds of contraction and/or expansion along coordinate axes in appropriate product of $p$-adic vector spaces (see Lemma 4.2).

**Remark.** In the case when all the numbers $\lambda_i, \mu_i$ are algebraic integers we have $S = \{\infty\}$ and Theorem 1.1 has much simpler formulation as all $p$-adic norms disappear. As an example consider the following expression
\[
\left(\sqrt{7} + 1\right)^n \left(3\sqrt{3} + 1\right)^m \xi_1 + \left(100\sqrt{5} + 3\right)^n \left(2\sqrt{2} + 1\right)^m \xi_2.
\]
(1.4)
It follows from Corollary 2.9 that the condition (iii) of Theorem 1.1 is satisfied and hence there is $\kappa \geq 1$ such that (1.4) is dense modulo $1/\kappa$.

**Remark.** A direct proof of Theorem 1.1 in case $S = \{\infty\}$ would be simpler. In this case the semigroup $\Sigma$ (constructed in §3) acts on $T^4 \times T^4$ instead of on the product of two solenoids. For the corresponding result concerning the action of a commutative semigroup of continuous endomorphisms on the $d$-dimensional torus see [4] (cf. [7]).

**Structure of the paper**

In §2 we state and prove some auxiliary results which are useful for deciding whether the given algebraic numbers $\lambda_i, \mu_i$ satisfy the assumptions of Theorem 1.1. In §3 we consider two commutative semigroups $\Sigma_1$ and $\Sigma_2$ of continuous endomorphisms of $\Omega^4_5$ and study the closed invariant sets for the corresponding
action of the diagonal subgroup of $\Sigma_1 \times \Sigma_2$ on the product $\Omega^4_a \times \Omega^4_a$. In §[14] we prove Theorem [11].

## 2. Examples

**Lemma 2.1.** Let $\lambda, \mu > 1$ be multiplicatively independent real algebraic numbers.

(i) Then for every positive integers $k \neq l$ the numbers $\lambda \mu$ and $\lambda^k \mu^l$ are multiplicatively independent.

(ii) Suppose that for some positive integers $k \neq l$, $\lambda \mu$ and $\lambda^k \mu^l$ are multiplicatively independent. Then $\lambda$ and $\mu$ are multiplicatively independent.

(iii) Suppose that $\deg Q \lambda = \deg Q \mu = 2$ and $\deg Q (\lambda \mu) = 4$. Then we have $Q(\lambda, \mu) = Q(\lambda \mu)$.

**Proof.** (i) and (ii) are obvious. We prove (iii). Since $\deg Q \lambda = \deg Q \mu = 2$, $[Q(\lambda, \mu) : Q] = [Q(\lambda, \mu), Q(\lambda)] [Q(\lambda), Q] = [Q(\lambda, \mu), Q(\lambda)] \cdot 2 \leq 4.$

By the assumption $\deg Q (\lambda \mu) = 4$, and so $[Q(\lambda \mu) : Q] = 4$.

Since $Q(\lambda \mu) \subset Q(\lambda, \mu)$, it follows that $[Q(\lambda, \mu) : Q] = 4$. □

The following lemmas will be used to check if a given pair of algebraic numbers is multiplicatively independent.

**Lemma 2.2.** Let $p, q > 1$ be the square-free numbers and $a, b \in Q$. If $p \neq q$, then $\sqrt{p} + a$ and $\sqrt{q} + b$ are multiplicatively independent.

**Proof.** Suppose that $\frac{\log(\sqrt{p} + a)}{\log(\sqrt{q} + b)} = w$ is rational and denote $w = \frac{r}{s}$. Then $(\sqrt{p} + a)^s = (\sqrt{q} + b)^r$. Since $Q(\sqrt{p}) \cap Q(\sqrt{q}) = Q$, it follows that $(\sqrt{p} + a)^s$ and $(\sqrt{q} + b)^r$ are rational. Consider $v = (\sqrt{p} + a)^s \in Q$. Let $\sigma$ be the automorphism of $Q(\sqrt{p})$ sending $\sqrt{p} \mapsto -\sqrt{p}$. Then $v = \sigma(v) = (\sqrt{q} + b)^r$ and consequently $u = \frac{\sqrt{p} + a}{\sqrt{p} + a}$ is an $r$th root of unity. Since $u \in Q(\sqrt{p})$, we must have $u = 1$ or $u = -1$, and we see that $u = -1$ and $a = 0$. Repeating the same argument with $v = (\sqrt{q} + b)^r$ we get that also $b = 0$. Therefore we have $q^w = p$. This is a contradiction. □

The following is an easy generalization of the previous result.

**Lemma 2.3.** Let $p, q > 1$ be the square-free numbers and $a, b, c, d \in Q$. If $p \neq q$, then $c\sqrt{p} + a$ and $d\sqrt{q} + b$ are multiplicatively independent.
Proof. Using the same argument as in the proof of the previous lemma we get,

\[(d\sqrt{q})^w = c\sqrt{p},\]

where \(w = \frac{r}{s} \in \mathbb{Q},\) and consequently

\[d^{2r}q^r = c^{2s}p^s.\]

Since at least one of \(r\) and \(s\) is odd we get a contradiction. \(\square\)

More generally, in [3, Proposition 4.2] it is proved that for \(\lambda\) and \(\mu\) which are effectively given complex algebraic numbers it is possible effectively to decide whether or not they are multiplicatively independent.

To decide if the condition (i) of Theorem 1.1 is satisfied we need to recall a result from [4]. Let \(k\) be a field. An algebraic number \(\beta\) over \(k\) is called torsion-free if \(\tilde{\beta}/\beta\) is not a root of unity for any \(\tilde{\beta} \neq \beta,\) where \(\tilde{\beta}\) and \(\beta\) are conjugate over \(k.\)

**Theorem 2.4** (Dubickas, [4, Theorem 2]). Suppose that \(\alpha\) is an algebraic number of degree \(d\) over a field \(k\) of characteristic zero, and let \(K\) be a normal closure of \(k(\alpha)\) over \(k.\) If \(\beta\) is torsion-free and \(L = k(\beta)\) is a normal extension of \(k\) of degree \(l\) and \(L \cap K = k,\) then

\[\deg_k(\alpha\beta) = dl.\]

**Remark 2.5.** It follows from Theorem 2.4 that if both \(\lambda_i’s\) and the \(\mu_i’s\) are of degree 2 and are not square roots of rational numbers and \(\lambda_i \notin \mathbb{Q}(\mu_i),\) then

\[\deg_{\mathbb{Q}}(\lambda_i^n\mu_i^m) = 4.\]

In the following lemmas we give conditions on \(\lambda_i, \mu_i\) that are easy to check and guarantee that (iii) of Theorem 1.1 holds.

Let \(S\) be as in Theorem 1.1. We denote \(S^* = S \setminus \{\infty\}.\) The following two lemmas are easy.

**Lemma 2.6.** Let \(\lambda_i, \mu_i, \ i = 1, 2,\) be real algebraic numbers of degree 2. Suppose that there exist positive integers \(k \neq l\) such that

\[\min_{p \in S^*} |\lambda_2\mu_2|_p > \max_{p \in S} |\lambda_1\mu_1|_p,\]

\[|\tilde{\lambda}_2\tilde{\mu}_2|_{\infty} > \max_{p \in S} |\lambda_1\mu_1|_p\]

and

\[\max_{p \in S} |\lambda_2\mu_2|_p < \min_{p \in S^*} |\lambda_1\mu_1|_p,\]

\[\max_{p \in S} |\lambda_2\mu_2|_p < |\tilde{\lambda}_1\tilde{\mu}_1|_{\infty}.\] (2.7)

Then there are \((\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0, 0)\}\) such that the inequalities (iii) of Theorem 1.1 hold.
LEMMA 2.8. Let $\lambda_i, \mu_i, i = 1, 2,$ be real algebraic numbers of degree 2. Suppose that either
\[
\max_{p \in S} |\lambda_2|_p < \min_{p \in S^*} |\lambda_1|_p,
\]
\[
\max_{p \in S^*} |\lambda_2|_p < \min_{p \in S^*} |\tilde{\lambda}_1|_p,
\]
\[
|\lambda_2|_\infty < |\tilde{\lambda}_1|_\infty
\]
or
\[
\min_{p \in S^*} |\mu_1|_p > \max_{p \in S} |\mu_2|_p,
\]
\[
|\tilde{\mu}_1|_\infty > \max_{p \in S} |\mu_2|_p.
\]
Then there exist positive integers $k \neq l$ such that (2.7) holds.

To sum up we have the following

COROLLARY 2.9. Let $\lambda_1, \mu_1$ and $\lambda_2, \mu_2$ be two distinct pairs of multiplicatively independent algebraic numbers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that the following conditions are satisfied:

- for every $n, m \in \mathbb{N}$, $\deg Q(\lambda_i^n \mu_i^m) = 4$,
- $|\lambda_i|_\infty > |\tilde{\lambda}_i|_\infty > 1$ and $|\mu_i|_\infty > |\tilde{\mu}_i|_\infty > 1$, $i = 1, 2$,
- $|\lambda_i|_p = |\tilde{\lambda}_i|_p > 1$ and $|\mu_i|_p = |\tilde{\mu}_i|_p > 1$, for $p \in S^* = S \setminus \{\infty\}$,
- $\min_{p \in S^*} |\lambda_2\mu_2|_p > \max_{p \in S} |\lambda_1\mu_1|_p$ and $|\tilde{\lambda}_2\tilde{\mu}_2|_\infty > \max_{p \in S} |\lambda_1\mu_1|_p$,
- $\max_{p \in S} |\lambda_2|_p < \min_{p \in S^*} |\lambda_1|_p$ and $|\lambda_2|_\infty < |\tilde{\lambda}_1|_\infty$

or
\[
\min_{p \in S^*} |\mu_1|_p > \max_{p \in S} |\mu_2|_p \quad \text{and} \quad |\tilde{\mu}_1|_\infty > \max_{p \in S} |\mu_2|_p.
\]

Then for any pair of real numbers $\xi_1, \xi_2$, with at least one $\xi_i$ non-zero, there exists a natural number $\kappa$ such that
\[
\{\lambda_1^n \mu_1^m \kappa \xi_1 + \lambda_2^n \mu_2^m \kappa \xi_2 : n, m \in \mathbb{N}\}
\]
is dense modulo 1.

3. The product semigroup $\Sigma$

In this section we construct a semigroup $\Sigma$ acting on the product of an appropriately chosen solenoid $\Omega_a$ and study the properties of this action. The semigroup $\Sigma$ will play an important role in the proof of Theorem 1.1. The idea of such a construction goes back to [3].
3.1. Definition of Σ

Let $\lambda_1, \mu_1$ and $\lambda_2, \mu_2$ be two distinct pairs of multiplicatively independent real algebraic numbers of degree 2, satisfying assumptions of Theorem 1.1. For positive integers $k \neq l$, define,

$$s_0 = \lambda_1 \mu_1, \quad r_0 = \lambda_2 \mu_2$$

(3.1) and

$$s_1 = \lambda_1^k \mu_1^l, \quad r_1 = \lambda_2^k \mu_2^l.$$  

(3.2)

By Lemma 2.1, $s_0, s_1$ and $r_0, r_1$ are multiplicatively independent and $\lambda_i^k \mu_i^l \in \mathbb{Q}(\lambda_i \mu_i)$. Therefore, we have that $s_1 \in \mathbb{Q}(s_0)$ and $r_1 \in \mathbb{Q}(r_0)$. Hence we can express the elements $s_1, r_1$ as polynomials with rational coefficients in $s_0$ and $r_0$, respectively,

$$s_1 = g(s_0), \quad r_1 = h(r_0),$$

(3.3)

where $g, h \in \mathbb{Q}[x]$. Let $N_0 = \mathbb{N} \cup \{0\}$. For $(\alpha, \beta) \in N_0^2$,

$$s_0^\alpha s_1^\beta = \lambda_1^{\alpha+k\beta} \mu_1^{\alpha+l\beta},$$

$$r_0^\alpha r_1^\beta = \lambda_2^{\alpha+k\beta} \mu_2^{\alpha+l\beta}.$$  

(3.4)

By the assumption (i) of Theorem 1.1

$$\deg_{\mathbb{Q}} s_0 = \deg_{\mathbb{Q}} s_1 = 4$$

(3.5)

and

$$\deg_{\mathbb{Q}} r_0 = \deg_{\mathbb{Q}} r_1 = 4.$$  

(3.6)

Let $\lambda > 1$ be a real algebraic number of degree $d$ with minimal (monic) polynomial $P_\lambda \in \mathbb{Q}[x]$,

$$P_\lambda(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_1 x + c_0.$$  

With $\lambda$ we associate the following companion matrix of $P_\lambda$,

$$\sigma_\lambda = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-c_0 & -c_1 & -c_2 & \ldots & -c_{d-1}
\end{pmatrix}.$$  

(3.7)

Let $\sigma_{s_0}$ and $\sigma_{r_0}$ be the companion matrices associated with $s_0$ and $r_0$, respectively. By (3.5) and (3.6), $\sigma_{s_0}, \sigma_{r_0} \in \text{GL}(4, \mathbb{Q})$. We put

$$\tau_{s_1} = g(\sigma_{s_0}) \quad \text{and} \quad \tau_{r_1} = h(\sigma_{r_0}),$$

(3.8)

where the polynomials $g$ and $h$ are defined in (3.3).
Let $\Sigma$ be the semigroup generated by $\left(\begin{array}{cc} \sigma_0 & 0 \\ 0 & \sigma_0 \end{array}\right)$ and $\left(\begin{array}{cc} 0 & \tau_{s_1} \\ \tau_{r_1} & 0 \end{array}\right)$, i.e.,

$$\Sigma = \left\langle \left(\begin{array}{cc} \sigma_0 & 0 \\ 0 & \sigma_0 \end{array}\right), \left(\begin{array}{cc} 0 & \tau_{s_1} \\ \tau_{r_1} & 0 \end{array}\right) \right\rangle.$$  

By (3.8) the semigroup $\Sigma$ is commutative. A general element of the semigroup $\Sigma$ is denoted by $\Omega$.

By a discrete subgroup $\Omega$ with the discrete topology. The dual group $\hat{\Omega}$ is the product of primes appearing in the denominators of the coefficients of the minimal polynomials $P_{\lambda_1}$ and $P_{\mu_1}$ of $\lambda_1, \mu_1$ (resp., the minimal polynomials $P_{\lambda_2}$ and $P_{\mu_2}$ of $\lambda_2, \mu_2$). Hence, $\Sigma_1$ (resp., $\Sigma_2$) is a finitely generated semigroup of continuous endomorphisms of a compact Abelian group $\Omega_{a_1}$ (resp., $\Omega_{a_2}$). Therefore, the set $\Sigma$ forms an Abelian semigroup of endomorphisms of $\Omega_{a_1} \times \Omega_{a_2}$, with the action given by

$$s^{(\alpha, \beta)}(x, y) = \left(s_1^{(\alpha, \beta)} x, s_2^{(\alpha, \beta)} y\right), \quad (x, y) \in \Omega_{a_1} \times \Omega_{a_2}.$$  

Let $a_1 = p_{i_1} p_{i_2} \cdots p_{i_{s_1}}$ and $a_2 = p_{j_1} p_{j_2} \cdots p_{j_{s_2}}$.

We denote

$$S_1 = \{p_{i_1}, p_{i_2}, \ldots, p_{i_{s_1}}\} \cup \{\infty\}.$$  

\footnote{Let $a = p_1 p_2 \ldots p_s$, where $p_i$ are different primes. Consider $\mathbb{Z}[1/a]$ as a topological group with the discrete topology. The dual group $\mathbb{Z}[1/a]$ of $\mathbb{Z}[1/a]$ is called an $a$-adic solenoid and we denote it by $\Omega$. The compact abelian group $\Omega_{a_1}$ may be considered as a quotient group of the additive group $\mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \cdots \times \mathbb{Q}_{p_s}^d$ by a discrete subgroup $B = \{(b, -b, \ldots, -b) : b \in \mathbb{Z}[1/a]^d\}$. That is, $\Omega_{a} = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d / B$. (For more details on solenoids see \cite{[5]}.)}
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\[ S_2 = \{ p_{j_1}, p_{j_2}, \ldots, p_{j_s} \} \cup \{ \infty \}. \]

Observe that the set \( S \) of primes defined in the statement of Theorem 1.1 is equal to

\[ S = S_1 \cup S_2. \]  \hspace{1cm} (3.9)

**Remark 3.10.** Let \( a \) be a product of different primes from the set \( S \setminus \{ \infty \} \). Then the semigroups \( \Sigma_i, i = 1, 2 \), act on \( \Omega^4_a \) and \( \Sigma \) acts on \( \Omega^4_a \times \Omega^4_a \).

### 3.2. Projected semigroups \( \Sigma_1 \) and \( \Sigma_2 \)

From now on the semigroups \( \Sigma_i \) act on \( \Omega^4_a \) and \( \Sigma \) acts on \( \Omega^4_a \times \Omega^4_a \), where \( a \) is defined in Remark 3.10.

We say that the semigroup \( \Sigma \) of continuous endomorphisms of a compact group \( G \) has the *ID-property*, or that \( \Sigma \) is an *ID-semigroup*, if the only infinite closed \( \Sigma \)-invariant \(^3\) subset of \( G \) is \( G \) itself.\(^4\)

Our aim in this subsection is to prove that \( \Sigma_1 \) and \( \Sigma_2 \) are ID-semigroups. We will use the following theorem which gives necessary and sufficient conditions in arithmetical terms for a commutative semigroup \( \Sigma \) of endomorphisms of \( \Omega^d_a \) to have the ID-property.

**Theorem 3.11** (Berend, [2, Theorem II.1]). A commutative semigroup \( \Sigma \) of continuous endomorphisms of \( \Omega^d_a \) has the ID-property if and only if the following hold:

(i) There exists an endomorphism \( \sigma \in \Sigma \) such that the characteristic polynomial \( f_{\sigma^n} \) of \( \sigma^n \) is irreducible over \( \mathbb{Q} \) for every positive integer \( n \).

(ii) For every common eigenvector \( v \) of \( \Sigma \) there exists an endomorphism \( \sigma_v \in \Sigma \) whose eigenvalue in the direction of \( v \) is of norm greater than 1.

(iii) \( \Sigma \) contains a pair of multiplicatively independent endomorphisms.\(^5\)

Let us explain in more details how to understand the statement of the condition (ii). Note that the semigroup \( \Sigma \) acts in a natural way on the “covering space” \( \mathbb{R}^d \times \mathbb{Q}^d_{p_1} \times \ldots \times \mathbb{Q}^d_{p_s} \). It is proved in [2] that the condition (i) implies that the roots \( \lambda_1, \ldots, \lambda_d, \sigma \) of \( \sigma \) are distinct and that there exists a basis \( v^{(i)} \in \mathbb{Q}(\lambda_{i, \sigma})^d \), \( i = 1, \ldots, d \), in which \( \Sigma \) has a diagonal form. Let \( K_j \) be the splitting field of the characteristic polynomial \( f_{\sigma} \) of \( \sigma \) over \( \mathbb{Q}_{p_j} \), \( j = 0, \ldots, s \), and let \( v^{1,j}, \ldots, v^{d,j} \) be a basis of \( K_j^d \) corresponding to \( v^{(i)} \), \( i = 1, \ldots, d \). The vectors \( v^{i,j} \), \( i = 1, \ldots, d \),

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\(^3\)Recall that a subset \( A \subset G \) is said to be \( \Sigma \)-invariant if \( \Sigma A \subset A \).

\(^4\)ID stands for *infinite invariant is dense*.

\(^5\)We say, as we do in the case of real numbers, that two endomorphisms \( \sigma \) and \( \tau \) are *rationally dependent* if there exist integers \( m \) and \( n \), not simultaneously equal to 0, such that \( \sigma^m = \tau^n \). Otherwise, we say that \( \sigma \) and \( \tau \) are *rationally independent*.
\( j = 0, \ldots, s \), are the common eigenvectors of \( \Sigma \). Denote by \( \lambda_{i,j,\tau}, i = 1, \ldots, d, \) the eigenvalues of any \( \tau \in \Sigma \), considered as a linear map of \( K_d^j \) with respect to the basis \( v^1,j, \ldots, v^d,j \). Then the condition (ii) says that for every \( 1 \leq i \leq d \) and \( 0 \leq j \leq s \) there exists a \( \sigma_{i,j} \in \Sigma \) such that \( |\lambda_{i,j,\sigma_{i,j}}|_{p_j} > 1 \).

**Lemma 3.12.** \( \Sigma_1 \) is a commutative ID-semigroup.

**Proof.** Commutativity of \( \Sigma_1 \) follows from (3.3) and definition (3.8) of \( \tau_{s_1} \). Now we have to check conditions of Theorem 3.11. Let us start with (iii). Since \( s_0 \) and \( s_1 \) are multiplicatively independent it follows that \( \sigma_{s_0} \) and \( \sigma_{s_1} \) are multiplicatively independent.

(ii) For a given matrix \( A \) we denote by \( \text{Spect}(A) \) the set of eigenvalues of \( A \). By the assumption, for \( p \in S, p \)-adic absolute values of \( \lambda_1, \mu_1 \) and their conjugates are greater than 1. Hence, by (3.1) and (3.2), it follows that

\[
\min\{|\rho|_p : \rho \in \text{Spect}(\sigma_{s_0}) \cup \text{Spect}(\tau_{s_1}), p \in S_1\} > 1.
\]

(i) Since, for every \( n, \deg_Q(s_{0n}) = 4 \) it follows that the characteristic polynomial \( f_{\sigma_{s_0}} \) of \( \sigma_{s_0} \) is irreducible over \( \mathbb{Q} \) for every \( n \in \mathbb{N} \). \( \square \)

**Lemma 3.13.** \( \Sigma_2 \) is a commutative ID-semigroup.

**Proof.** Analogous to the proof of Lemma 3.12. \( \square \)

For a given subset \( A \subset \Omega_4^4 \times \Omega_4^4 \) and \( \omega_1, \omega_2 \in \Omega_4^4 \), we define

\[
A_{\omega_1} = \{\omega_2 \in \Omega_4^4 : (\omega_1, \omega_2) \in A\} \subset \Omega_4^4, \\
A_{\omega_2} = \{\omega_1 \in \Omega_4^4 : (\omega_1, \omega_2) \in A\} \subset \Omega_4^4.
\]

The following lemma follows immediately from Lemmas 3.12 and 3.13.

**Lemma 3.15.** Let \( A \) be a non-empty, \( s^{(1,0)} \)- and \( s^{(0,1)} \)-invariant closed subset of \( \Omega_4^4 \times \Omega_4^4 \). Then

(i) the set \( P_2 = \{\omega_2 \in \Omega_4^4 : A_{\omega_2} \neq \emptyset\} \) is either the whole \( \Omega_4^4 \) or is a finite set of torsion elements in \( \Omega_4^4 \);

(ii) the set \( P_1 = \{\omega_1 \in \Omega_4^4 : A_{\omega_1} \neq \emptyset\} \) is either the whole \( \Omega_4^4 \) or is a finite set of torsion elements in \( \Omega_4^4 \).

For a given positive integer \( q \) we denote by \( \Omega_a^d(q) \) the subgroup of \( \Omega_a^d \) consisting of all elements whose order divides \( q \). It is known (see [2] Lemma II.13) that for every \( q \in \mathbb{N} \), the subgroup

\[
\Omega_a^d(q) \simeq \mathbb{Z}[1/a]^d/q\mathbb{Z}[1/a]^d \simeq (\mathbb{Z}[1/a]/q\mathbb{Z}[1/a])^d
\]

is finite.
\textbf{Lemma 3.16 ([N Lemma 3.2]).} Let \( \sigma \) be a \( d \times d \)-invertible matrix with entries from the ring \( \mathbb{Z}[1/a] \). Let \( r \in \Omega_a^d(q) \subset \Omega_a^d, q = q_1^{\alpha_1}q_2^{\alpha_2}\ldots q_m^{\alpha_m} \), where \( q_i \) are different primes and \( \alpha_i \in \mathbb{N} \), be a torsion element. Assume that for \( 1 \leq i \leq m \), \( |\det \sigma|_{q_i} = 1 \). Then there exists a \( k \in \mathbb{N} \) such that
\[
\sigma^k r = r \text{ in } \Omega_a^d.
\]

\textbf{Lemma 3.17.} Let \( A \) be a non-empty, \( s^{(1,0)} \)– and \( s^{(0,1)} \)-invariant closed subset of \( \Omega_a^4 \times \Omega_a^4 \). Then \( A \) contains a torsion element.

\textbf{Remark.} Although Lemma 3.17 can be deduced from appropriate analogues of [N Lemma 3.3 and Lemma 3.4] and their proofs, we decided to present here a detailed proof as it is one of the crucial results.

\textbf{Proof.} By Lemma 3.15 the set \( P_2 = \{ \omega_2 \in \Omega_a^4 : A_{\omega_2} \neq \emptyset \} \) is either the whole \( \Omega_a^4 \) or is a finite set of torsion elements in \( \Omega_a^4 \). There are two cases. The first one, when \( P_2 \) contains a torsion element \( \omega_2 \) of degree \( q = q_1^{\alpha_1}\ldots q_r^{\alpha_r} \) such that
\[
|\det s_1^{(1,0)}|_{q_j} = |\det s_1^{(0,1)}|_{q_j} = 1, \quad j = 1, \ldots, r,
\]
(in particular this holds if \( P_2 = \Omega_a^4 \)). Then, by Lemma 3.16 we can find \( k_1 \) and \( k_2 \in \mathbb{N} \) such that \( (s_1^{(1,0)})^{k_1} \omega_2 = \omega_2 \) and \( (s_1^{(0,1)})^{k_2} \omega_2 = \omega_2 \). Thus we see that \( A_{\omega_2} \) is a non-empty, closed, \( (s_1^{(1,0)})^{k_1} \)- and \( (s_1^{(0,1)})^{k_2} \)-invariant subset of \( \Omega_a^4 \). Clearly, the semigroup \( \langle (s_1^{(1,0)})^{k_1}, (s_1^{(0,1)})^{k_2} \rangle \) is an ID-semigroup of endomorphisms of \( \Omega_a^4 \) (see the proof of Lemma 3.12). Hence, \( A_{\omega_2} \) is either a finite set of torsion elements or is the whole \( \Omega_a^4 \). Thus, the lemma follows in this case.

Consider the second case. If \( P_2 \) is a finite set of torsion elements but there is no torsion element of degree \( q = q_1^{\alpha_1}\ldots q_r^{\alpha_r} \) such that
\[
|\det s_1^{(1,0)}|_{q_j} = |\det s_1^{(0,1)}|_{q_j} = 1, \quad j = 1, \ldots, r,
\]
then we pick up arbitrary torsion element \( \omega_2 \) from \( P_2 \), and instead of \( A \) we consider the set \( A' \) which is obtained from \( A \) by multiplying the second coordinate by the order of \( \omega_2 \). Then the set \( P_2 \) corresponding to \( A' \) contains 0. Since \( A_{\omega_2} \) was non-empty, the set
\[
A_0' = \{ \omega_1 \in \Omega_a^4 : (\omega_1, 0) \in A' \} \subset \Omega_a^4
\]
is also non-empty. Moreover, since \( A' \) is \( s^{(1,0)} \)– and \( s^{(0,1)} \)-invariant it follows that \( A_0' \) is \( s_1^{(1,0)} \)- and \( s_1^{(0,1)} \)-invariant. Finally, it is clear that \( A_0' \) is closed. Since the semigroup \( \langle (s_1^{(1,0)}), (s_1^{(0,1)}) \rangle \) is an ID-semigroup of endomorphisms of \( \Omega_a^4 \) (Lemma 3.12), it follows that \( A_0' \) is either a finite set of torsion elements or is the whole \( \Omega_a^4 \). Thus, we have proved that \( A' \) contains a torsion element. But then also \( A \) must contain a torsion element and the lemma follows. \( \square \)
Remak. Notice that we do not have to use Lemma 3.16 in the proof of Lemma 3.17. In fact, in both cases we can proceed as in the second case. We included the reasoning with Lemma 3.16 since it shows a nice property that for the torsion element $\omega_2$ satisfying assumptions of Lemma 3.16, the set $A_{\omega_2}$ is either a finite set of torsion elements or is the whole $\Omega_4^a$.

Lemma 3.18. Let $A$ be a closed, $s^{(1,0)}$- and $s^{(0,1)}$-invariant subset of $\Omega_4^a \times \Omega_4^a$. If all torsion elements of $A$ are isolated in $A$, then $A$ is finite.

Proof. The same as the proof of [8, Lemma 3.4]. □

4. Proof of Theorem 1.1

Let $a$ be the product of different primes appearing in $a_1$ and $a_2$, that is

$$a = \prod_{p \in S \setminus \{\infty\}} p,$$

where $S$ is defined in (3.9). As we observed there

$$S = \{\infty, p_1, p_2, \ldots, p_s\}$$

with $p_i$’s as in Theorem 1.1.

For $\omega = (\omega_1, \omega_2) \in \Omega_4^a \times \Omega_4^a$ consider the orbit $\Sigma \omega$ of the point $\omega$ under the action of the semigroup $\Sigma = \langle s^{(1,0)}, s^{(0,1)} \rangle$,

$$\Sigma \omega = \left\{ (s_1^{(\alpha, \beta)} \omega_1, s_2^{(\alpha, \beta)} \omega_2) \in \Omega_4^a \times \Omega_4^a : (\alpha, \beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\}\right\}. \quad (4.1)$$

Clearly, $\Sigma \omega$ is $\Sigma$-invariant.

Consider a general element $s^{(\alpha, \beta)}$ of the semigroup $\Sigma$,

$$s^{(\alpha, \beta)} = \begin{pmatrix} \sigma_{s_0}^\alpha & \tau_{s_1}^\beta & 0 & 0 \\ 0 & \sigma_{s_0}^\alpha & \tau_{s_1}^\beta & 0 \end{pmatrix}, \quad \text{for some } (\alpha, \beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\}.$$ 

Denote the diagonal elements of the matrix $s^{(\alpha, \beta)}$, which are $4 \times 4$-nonsingular matrices belonging to $M(4, \mathbb{Z}[1/a])$, by $s_1^{(\alpha, \beta)}$ and $s_2^{(\alpha, \beta)}$. By definition

$$s_1^{(\alpha, \beta)} = \sigma_{s_0}^\alpha \tau_{s_1}^\beta \in M(4, \mathbb{Z}[1/a_1])$$

and

$$s_2^{(\alpha, \beta)} = \sigma_{s_0}^\alpha \tau_{s_1}^\beta \in M(4, \mathbb{Z}[1/a_2]).$$

We denote the conjugates of $s_0$ and $s_1$ as follows

$$s_0 = s_{0,1} = \lambda_1 \mu_1, \quad s_{0,2} = \lambda_1 \tilde{\mu}_1, \quad s_{0,3} = \lambda_1 \bar{\mu}_1, \quad s_{0,4} = \lambda_1 \bar{\mu}_1,$$

$$s_1 = s_{1,1} = \lambda_1^k \mu_1^l, \quad s_{1,2} = \lambda_1^k \tilde{\mu}_1^l, \quad s_{1,3} = \lambda_1^k \bar{\mu}_1^l, \quad s_{1,4} = \lambda_1^k \bar{\mu}_1^l.$$
and similarly for $r_0$ and $r_1$,
\[
\begin{align*}
  r_0 &= r_{0,1} = \lambda_2 \mu_2, & r_{0,2} &= \tilde{\lambda}_2 \mu_2, & r_{0,3} &= \lambda_2 \tilde{\mu}_2, & r_{0,4} &= \tilde{\lambda}_2 \tilde{\mu}_2, \\
  r_1 &= r_{1,1} = \lambda_k \mu_l, & r_{1,2} &= \tilde{\lambda}_k \mu_l, & r_{1,3} &= \lambda_k \tilde{\mu}_l, & r_{1,4} &= \tilde{\lambda}_k \tilde{\mu}_l.
\end{align*}
\]
For $p \in S$, let $\rho_{p,1} \geq \rho'_{p,1}$ (resp., $\rho_{p,2} \geq \rho'_{p,2}$) denote the $p$-adic norms of the maximum and minimum of the eigenvalues of the matrix $s_1^{(\alpha, \beta)}$ (resp., $s_2^{(\alpha, \beta)}$).

We have
\[
\begin{align*}
  \rho_{p,1} &= \max \left\{ |s_{0,i}|_p |s_{1,j}|_p : i, j = 1, 2, 3, 4 \right\}, \\
  \rho'_{p,1} &= \min \left\{ |s_{0,i}|_p |s_{1,j}|_p : i, j = 1, 2, 3, 4 \right\}
\end{align*}
\]
and
\[
\begin{align*}
  \rho_{p,2} &= \max \left\{ |r_{0,i}|_p |r_{1,j}|_p : i, j = 1, 2, 3, 4 \right\}, \\
  \rho'_{p,2} &= \min \left\{ |r_{0,i}|_p |r_{1,j}|_p : i, j = 1, 2, 3, 4 \right\}.
\end{align*}
\]
It is easy to check that we have the following

**Lemma 4.2.** Under the assumptions of Theorem [1.1] there exists $(\alpha_0, \beta_0)$ such that $s^{(\alpha_0, \beta_0)}$ satisfies
\[
\min_{p \in S} \rho'_{p,2} > \max_{p \in S} \rho_{p,1} > 1. \tag{4.3}
\]
Moreover, there exists $(\alpha'_0, \beta'_0)$ such that $s^{(\alpha'_0, \beta'_0)}$ satisfies
\[
\min_{p \in S} \rho'_{p,1} > \max_{p \in S} \rho_{p,2} > 1. \tag{4.4}
\]

We introduce the following notation. Let
\[
V_1 \times V_2 := \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4 \times \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4,
\]
For $i = 1, 2$ we write $V_i = (V_{i,0}, V_{i,1}, \ldots, V_{i,s})$, where $V_{i,j} = \mathbb{Q}_{p_j}^4$. By $\mathbb{Q}_{a}^4$ we denote the "covering space" of $\Omega_{a}^4$, i.e.,
\[
\mathbb{Q}_{a}^4 = \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4.
\]
If the spaces $\mathbb{Q}_{p_j}^4$, $j = 0, \ldots, s$, are equipped with norms $\| \cdot \|_{p_j}$, then, for
\[
z = (z_0, z_1, \ldots, z_s) \in \mathbb{Q}_{a}^4,
\]
we put
\[
\|z\|_{\mathbb{Q}_{a}^4} = \max_{0 \leq j \leq s} \|z_j\|_{p_j}.
\]
The space $\mathbb{Q}^4_a$ becomes a metric space with the distance

$$d_{\mathbb{Q}^4_a}(z, w) = \|z - w\|_{\mathbb{Q}^4_a}.$$ 

Let

$$\pi : \mathbb{Q}^4_a \to \Omega^4_a = \mathbb{Q}^4_a / B$$

be the canonical projection, i.e., $\pi(z) = z + B$, where

$$B = \left\{(b, -b, \ldots, -b) : b \in \mathbb{Z}[1/a]^4\right\},$$

is a closed discrete subgroup of $\mathbb{Q}^4_a$.

If $b = (b, -b, \ldots, -b) \in B$, we denote its coordinates by $b_j$, $0 \leq j \leq s$, i.e.,

$$b_0 = b \quad \text{and} \quad b_j = -b \quad \text{for} \quad j = 1, \ldots, s.$$ 

It is easy to check that the following function,

$$d_{\Omega^4_a}(z + B, w + B) = \inf_{b \in B} d_{\mathbb{Q}^4_a}(z - b, w - b)$$

$$= \inf_{b \in B} \|z - w - b\|_{\mathbb{Q}^4_a}$$

$$= \inf_{b \in B} \max_{0 \leq j \leq s} \|z_j - w_j - b_j\|_{p_j},$$

(4.5)

defines the metric on $\Omega^4_a$.

The vector $(1, s_0, s_0^2, s_0^3)^t$ is an eigenvector of the matrix $s_1^{(1,0)}$ with an eigenvalue $s_0$, that is

$$s_1^{(1,0)} (1, s_0, s_0^2, s_0^3)^t = s_0 (1, s_0, s_0^2, s_0^3)^t \in \mathbb{R}^4.$$

Since $\Sigma_1$ is a commutative semigroup it follows that

$$v = (1, s_0, s_0^2, s_0^3)^t \in \mathbb{R}^4 \times \mathbb{Q}_p^4 \times \cdots \times \mathbb{Q}_p^4$$

is a common eigenvector of $\Sigma_1$ acting on $\mathbb{R}^4 \times \mathbb{Q}_p^4 \times \cdots \times \mathbb{Q}_p^4$. In particular,

$$s_1^{(1,0)} v = s_0 v \quad \text{and} \quad s_1^{(0,1)} v = \tau_{s_1} v = g(s_0) v = g(s_0) v = s_1 v.$$ 

(4.6)

Similarly, the vector $(1, r_0, r_0^2, r_0^3)^t$ is an eigenvector of the matrix $\sigma_{r_0}$ with an eigenvalue $r_0$. Since $\tau_{r_1} = h(\sigma_{r_0})$ for $h \in \mathbb{Q}[x]$ with $r_1 = h(r_0)$, we get that for the vector

$$w = (1, r_0, r_0^2, r_0^3)^t \in \mathbb{R}^4 \times \mathbb{Q}_p^4 \times \cdots \times \mathbb{Q}_p^4$$

we have

$$\sigma_{r_0}^{\alpha} \tau_{r_1}^\beta w = r_0^\alpha r_1^\beta w.$$ 

(4.7)
Let $\xi_1$ and $\xi_2$ be two non-zero real numbers. We set

$$\omega_0 = (v\xi_1, w\xi_2) \in \prod_{p \in S} \mathbb{Q}_p^4 \times \prod_{p \in S} \mathbb{Q}_p^4.$$  \hfill (4.8)

By (4.6) and (4.7) we have,

$$s^{(\alpha, \beta)}(\omega_0) = \pi \left( s_0^{\alpha_1} \xi_1, s_0^{\alpha_2} s_1^{\alpha_3} \xi_1, s_0^{\alpha_4} s_1^{\alpha_5} \xi_1, 0, \ldots, 0, \right. \left. r_0^{\alpha_6} r_1^{\alpha_7} \xi_2, r_0^{\alpha_8} r_1^{\alpha_9} \xi_2, r_0^{\alpha_{10}} r_1^{\alpha_{11}} \xi_2, 0, \ldots, 0 \right).$$  \hfill (4.9)

We define a homomorphism $\chi_d : \Omega_a^d \to T^d$. Let $\chi_1 : \Omega_a = R \times \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p / B \to T$, be given by\footnote{Every $x \in \mathbb{Q}_p$ can be uniquely expressed as a convergent, in $| \cdot |_p$-norm, sum (Hensel representation), $x = \sum_{k=1}^{\infty} x_k p^k$, for some $t \in \mathbb{Z}$ and $x_k \in \{0, 1, \ldots, p - 1\}$. The fractional part of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, is 0 if the number $t$ in the Hensel representation is greater than or equal to 0, and equal to $\sum_{k=0}^{t-1} x_k p^k$, if $t < 0$.}

$$\chi_1 \left( (x_0, x_1, \ldots, x_s) + B \right) = e^{2\pi i x_0} e^{2\pi i \{x_1\}_p} \cdots e^{2\pi i \{x_s\}_p}. $$

Since $x \mapsto e^{2\pi i \{x\}_p}$ is a homomorphism from $\mathbb{Q}_p$ to the 1-torus $T = \mathbb{R}/\mathbb{Z}$, it is easy to check that the map $\chi_1$ is well defined, i.e., for every $r \in \mathbb{Z}[1/a]$, we have

$$\chi_1 \left( (x_0 + r, x_1 - r, \ldots, x_s - r) + B \right) = \chi_1 \left( (x_0, x_1, \ldots, x_s) + B \right).$$

Now, we extend the map $\chi_1$ to $\Omega_a^d$, $d > 1$. For $j = 0, \ldots, s$, we denote

$$x^j = (x_1^j, \ldots, x_d^j) \in \mathbb{Q}_p^d.$$ 

Now we define a homomorphism $\chi_d : \Omega_a^d \to T^d$ by the formula

$$\chi_d \left( (x_0^j, x_1^j, \ldots, x_s^j) + B^d \right) = \chi_1 \left( (x_1^j, x_1^j, \ldots, x_s^j) + B \right) \ldots, \chi_1 \left( (x_d^j, x_d^j, \ldots, x_s^j) + B \right).$$  \hfill (4.10)

**Lemma 4.11.** Let $\Omega$ be the set of accumulation points of the $\Sigma$-orbit of $\pi(\omega_0)$. If $(0, 0) \in \Omega$ then one of the following holds:

1. the point $(0, 0)$ is isolated in $\Omega$,
2. the set $\Omega$ contains at least one of the following sets

$$T_1 = \Omega_a^4 \times \{0\},$$

$$T_2 = \{0\} \times \Omega_a^4.$$  \hfill (4.12)
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**Remark 4.13.** Let $F$ be a close infinite $\Sigma$-invariant subset of $\Omega^4_a \times \Omega^4_a$, and let $F^{ac}$ denote the set of accumulation points of $F$. It will be clear from the proof that Lemma 4.11 is valid for $F^{ac}$ in place of $\Omega$.

We postpone the proof of Lemma 4.11 to the next subsection and continue with the proof of Theorem 1.1.

First we note that immediately from Lemma 4.11 we get the following

**Corollary 4.14.** If $(0,0) \in \Omega$ then one of the following holds:

1. the point $(0,0)$ is isolated in $\Omega$,
2. the set $\Omega^* = \{\omega_1 + \omega_2 : \omega = (\omega_1,\omega_2) \in \Omega\}$ is equal to $\Omega^4_a$.

**Proof of Theorem 1.1** We can assume that both $\xi_1$ and $\xi_2$ are non-zero; if one of them is zero then Theorem 1.1 follows by result of [3]. Consider the set $\Omega$. Assume first that $(0,0)$ is not isolated in $\Omega$. Notice that by (3.4) it follows that the first and the 4th + 5th coordinate of (4.9) are equal to
\[ \lambda_1^{\alpha+\beta k} \mu_1^{\alpha+\beta l} \xi_1 \quad \text{and} \quad \lambda_2^{\alpha+\beta k} \mu_1^{\alpha+\beta l} \xi_2. \]

Hence, density modulo 1 of the set \( \left\{ \lambda_1^{\alpha+\beta k} \mu_1^{\alpha+\beta l} \xi_1 + \lambda_2^{\alpha+\beta k} \mu_1^{\alpha+\beta l} \xi_2 : (\alpha, \beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\} \right\} \) (4.15) follows from Corollary 4.14 if we take the image of the set $\Omega^* = \Omega^4_a$ by the map $\chi_4$ (4.10) composed with the projection on the first coordinate in $T^4$. But (4.15) is a subset of (1.2). (Note that in this case $\kappa = 1$.)

Next suppose that $(0,0)$ is isolated. By Lemma 3.17 and Lemma 3.18 there is a non-isolated torsion element $(q_2, q_2)^t \in \Omega$. Proceeding as in the proof of [8, Theorem 1.5], that is multiplying by the matrix $\kappa \text{Id}$, where $\kappa q_1 = \kappa q_2 = 0$, and using Remark 4.13 we get that $\kappa \Omega^* = \Omega^4_a$. Hence, the theorem is proved. □

**4.1. Proof of Lemma 4.11**

The proof follows the main steps of [8], where the product of the 2-dimensional solenoids was considered. In the proof we will need the following lemma. Let $k$ be a local field (in our case $k = \mathbb{R}$ or the finite extension of $\mathbb{Q}_p$), equipped with an absolute value $|\cdot|$ (if $|\cdot| = |\cdot|_\infty$ or $|\cdot|_p$, resp.), $K$ be an algebraic closure of $k$. The unique extension of $|\cdot|$ to the absolute value on $K$ will also be denoted by $|\cdot|$.

**Lemma 4.16.** Let $A \in \text{GL}(2,k)$. Suppose that $A$ has two different eigenvalues $\eta_1, \eta_2 \in K$, such that $|\eta_1| > |\eta_2| > 1$. Then there exists a norm $\|\cdot\|$ in $k^2$ such that

1. $\|A\| = |\eta_1|$ and $\|A^{-1}\| = \frac{1}{|\eta_2|};$
2. $\|Av\| \geq |\eta_2||v||$ and $\|A^{-1}v\| \geq \frac{1}{|\eta_1||v||}$.
Lemma 4.2 there is $(\alpha_1, \beta_1)$ metrics $d_1$ metric it is non-zero (infinite, resp.) for all equivalent metrics. In particular, if the limit (4.18) is non-zero (infinite, resp.) for one \(\Omega\). Since all norms on finite dimensional vector space are equivalent, the Proof of Lemma 4.11. Let $\text{pr}_i$ $(i = 1, 2)$ be the projection from the product $\Omega_a^4 \times \Omega_a^4$ onto its first and second “coordinate”, respectively. By Lemma 3.12 the semigroup $\Sigma_1 = \langle s_1^{(1,0)}, s_1^{(0,1)} \rangle$ is an ID-semigroup. Since $s_1$ is irrational \(\pi_1(\pi(v))\) is not a torsion point. Hence, we obtain that for every $\omega_1 \in \Omega_a^4$ there exist sequences $\{\alpha_k\}$ and $\{\beta_k\}$, tending to infinity, such that

$$s_1^{(\alpha_k, \beta_k)} \text{pr}_1(\pi(v)) = \left(s_1^{(1,0)}\right)^{\alpha_k} \left(s_1^{(0,1)}\right)^{\beta_k} \text{pr}_1(\pi(v)) \to \omega_1,$$

as $k \to \infty$. Since $\Omega_a^4$ is compact, we can assume, choosing a subsequence, that there exists $\omega_2 \in \Omega_a^4$ such that $s_2^{(\alpha_k', \beta_k')} \text{pr}_2(\pi(w)) \to \omega_2$. Therefore, for every $\omega_1 \in \Omega_a^4$ there exists $\omega_2 \in \Omega_a^4$ so that $(\omega_1, \omega_2) \in \Omega$. In particular, we see that $\Omega$ is infinite.

Clearly, $\Omega$ is a non-empty, $s^{(1,0)}$- and $s^{(0,1)}$-invariant closed subset of $\Omega_a^4 \times \Omega_a^4$. By Lemma 3.15 the intersection of $\Omega$ with the “axes” $T_1$ and $T_2$ either is empty, contains finitely many torsion elements, or equals $T_i$, $i = 1, 2$. Assume that for $i = 1, 2$,

$$\Omega \cap T_i$$

is empty or a finite set of torsion elements. (4.17)

We will show that this assumption leads to contradiction if $(0, 0)$ is not isolated in $\Omega$.

Since $(0, 0) \in \Omega_a^4 \times \Omega_a^4$ is not isolated in $\Omega$ there exists a sequence $\{(x_n + B, y_n + B)\} \subset \Omega$ tending to $(0, 0)$. By (4.17) it follows that $x_n + B \neq 0$ and $y_n + B \neq 0$. Without loss of generality, choosing an appropriate representative from $x_n + B$ ($y_n + B$, resp.), we can assume that $x_n \neq 0$, $\|x_n\|_{Q_a^4} \to 0$, and $y_n \neq 0$, $\|y_n\|_{Q_a^4} \to 0$. Choosing an appropriate subsequence, we can assume that

$$\lim_{n \to \infty} \frac{d_{11_a}^2(y_n + B, 0)}{d_{11_a}^2(x_n + B, 0)} = c \in [0, +\infty].$$

(4.18)

**Remark.** Since all norms on finite dimensional vector space are equivalent, the metrics $d_{11_a}^2$ is defined with the use of different norms on the covering space, are equivalent. In particular, if the limit (4.18) is non-zero (infinite, resp.) for one metric it is non-zero (infinite, resp.) for all equivalent metrics.

Consider the case when $c \neq 0$ or the limit in (4.18) is infinite. By (4.3) of Lemma 4.2 there is $\alpha_0, \beta_0) \in N_2^4 \setminus \{(0, 0)\}$ such that the element $s^{(\alpha_0, \beta_0)}$ satisfies

$$\min_{p \in S} \rho_{p, 2} > \max_{p \in S} \rho_{p, 1} > 1.$$  (4.19)
In what follows we fix such an element \( s^{(\alpha_0, \beta_0)} \). By Lemma 4.16 we have norms \( \| \cdot \|_{p,j,i} \) in \( V_i,j \) such that, for every \( y \in V_{2,j} \),

\[
\rho_{p,j,2} \| y \|_{p,j,2} \geq \| s^{(\alpha, \beta)}_2(y) \|_{p,j,2} \geq \rho'_{p,j,2} \| y \|_{p,j,2};
\]

(4.20)

and for every \( x \) in \( V_{1,j} \) we have,

\[
\| s^{(\alpha, \beta)}_1(x) \|_{p,j,1} \leq \rho_{p,j,1} \| x \|_{p,j,1}.
\]

(4.21)

We consider the product \( \Omega^4_a \times \Omega^4_a \) endowed with \( (d_{\Omega^4_a}^1, d_{\Omega^4_a}^2) \), where \( d_{\Omega^4_a}^i \)'s are defined by \( (4.5) \) with the use of the norms defined in \( (4.20) \) for \( i = 2 \), and \( (4.21) \) for \( i = 1 \).

**Lemma 4.22.** Let \( s^{(\alpha_0, \beta_0)} \) be as above. Then there exist constants \( \rho > 1 \) and \( 1 > \gamma > 0 \) such that for every \( y + B \in \Omega^4_a \) satisfying \( d_{\Omega^4_a}^2(y + B, 0) < \frac{\gamma}{2} \), we have

\[
d_{\Omega^4_a}^2(s^{(\alpha_0, \beta_0)}_2(y + B), 0) \geq \rho d_{\Omega^4_a}^2(y + B, 0).
\]

In particular, iterating the above inequality, it follows that \( s^{(\alpha_0, \beta_0)}_2 \) is expansive, \( i.e. \), there exists an open ball \( U \) (of radius \( \gamma/2 \)) around 0 such that for every

\[
0 \neq y + B \in U,
\]

there exists \( l \) such that

\[
(s^{(\alpha_0, \beta_0)}_2)^l y + B \notin U.
\]

**Proof.** Let \( \varepsilon = \inf_{b \in B \setminus \{0\}} \| b \|_{Q^4_a} \), \( \gamma = \frac{\varepsilon}{\max_{p \in S} \rho'_{p,2}} \), and \( \rho = \min_{p \in S} \rho'_{p,2} \).

Changing the representative \( y \), if necessary, we can assume that

\[
\| y \|_{Q^4_a} < \gamma/2.
\]

(4.23)

For simplicity, we denote the matrix \( s^{(\alpha_0, \beta_0)}_2 \) by \( A \). By \( (4.20) \) and \( (4.21) \) we get,

\[
d_{\Omega^4_a}^2(A(y + B), 0) = \inf_{b \in B} \max_{0 \leq j \leq s} \| Ay_j - b_j \|_{p,j,2}
\]

\[
= \inf_{b \in B} \max_{0 \leq j \leq s} \| A(y_j - A^{-1}b_j) \|_{p,j,2}
\]

\[
\geq \inf_{b \in B} \max_{0 \leq j \leq s} \rho'_{p,j,2} \| y_j - A^{-1}b_j \|_{p,j,2}
\]

\[
\geq \inf_{b \in B} \max_{0 \leq j \leq s} \min_{p \in S} \rho'_{p,j,2} \| y_j - A^{-1}b_j \|_{p,j,2}
\]

\[
= \rho \inf_{b \in B} \| y - A^{-1}b \|_{Q^4_a}.
\]

(4.24)

It follows from Lemma 4.16 and \( (4.20) \) that for every non-zero element

\[
b = (b, -b, \ldots, -b) \in B,
\]

(4.25)
we have
\[ \| A^{-1} b \|_{Q_4} = \max_{0 \leq j \leq s} \| A^{-1} b \|_{p_j, 2} \geq \frac{1}{\max_{0 \leq j \leq s} \rho_{p_j, 2}} \| b \|_{p_j, 2} \]
\[ \geq \frac{1}{\max_{p \in S} \rho_{p, 2}} \| b \|_{Q_4} \geq \frac{1}{\max_{p \in S} \rho_{p, 2}} \varepsilon = \gamma. \] (4.25)

By (4.23) and (4.25) we get
\[ \inf_{b \in B} \| y - A^{-1} b \|_{Q_4} = \inf_{b \in B} \| y - b \|_{Q_4} = \| y \|_{Q_4} = d_{\Omega^4}(y + B, 0). \] (4.26)

Now (4.26) and (4.24) imply the conclusion. \[ \square \]

For every \( l \in \mathbb{N} \),
\[ d_{\Omega^4}^l \left( \left( s_{1}^{(\alpha_0, \beta_0)} \right)^l (x_n + B), 0 \right) = \inf_{b \in B} \max_{0 \leq j \leq s} \left\| \left( s_{1}^{(\alpha_0, \beta_0)} \right)^l (x_n)_j - b_j \right\|_{p_j, 1} \]
\[ = \inf_{b \in B} \max_{0 \leq j \leq s} \left\| \left( s_{1}^{(\alpha_0, \beta_0)} \right)^l \left( (x_n)_j - \left( s_{1}^{(\alpha_0, \beta_0)} \right)^{-l} b_j \right) \right\|_{p_j, 1} \]
\[ \leq \inf_{b \in B} \max_{0 \leq j \leq s} (\rho_{p_j, 1})^l \left\| (x_n)_j - \left( s_{1}^{(\alpha_0, \beta_0)} \right)^{-l} b_j \right\|_{p_j, 1} \]
\[ \leq \left( \max_{p \in S} \rho_{p, 1} \right)^l \inf_{b \in B} \left\| x_n - \left( s_{1}^{(\alpha_0, \beta_0)} \right)^{-l} b \right\|_{Q_4} \]
\[ \leq \left( \max_{p \in S} \rho_{p, 1} \right)^l \| x_n \|_{Q_4}. \]

Since \( \| x_n \|_{Q_4} \to 0 \), we can assume that \( \| x_n \|_{Q_4} < \frac{1}{2} \inf_{b \in B \setminus \{ 0 \}} \| b \|_{Q_4} \). Then
\[ \| x_n \|_{Q_4} = d_{\Omega^4}^1 (x_n + B, 0), \]
and consequently we have
\[ d_{\Omega^4}^l \left( \left( s_{1}^{(\alpha_0, \beta_0)} \right)^l (x_n + B), 0 \right) \leq \left( \max_{p \in S} \rho_{p, 1} \right)^l \] (4.27)
\[ \leq \left( \max_{p \in S} \rho_{p, 1} \right)^l d_{\Omega^4}^1 (x_n + B, 0). \]

Similarly,
\[ d_{\Omega^4}^2 \left( \left( s_{1}^{(\alpha_0, \beta_0)} \right)^l (y_n + B), 0 \right) \leq \left( \max_{p \in S} \rho_{p, 2} \right)^2 d_{\Omega^4}^2 (y_n + B, 0). \] (4.28)

Fix \( r \) such that \( 1 / (\max_{p \in S} \rho_{p, 2})^r \leq \gamma / 2 \). Since \( \| y_n \|_{Q_4} \to 0 \) we may assume that
\[ \| y_n \|_{Q_4} \leq \varepsilon / 2 \quad \text{and} \quad d_{\Omega^4}^2 (y_n + B, 0) \leq 1 / \left( \max_{p \in S} \rho_{p, 2} \right)^r. \]
By Lemma 4.22 and (4.28), for each \( n \), there exists the smallest number \( l_n \in \mathbb{N} \) such that

\[
d_{\Omega_4}^2 \left( \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n} (y_n + B), 0 \right) \geq \rho^{l_n} d_{\Omega_4}^2 (y_n + B, 0), \quad \text{where} \quad \rho = \min_{\rho' \in S} \rho'_{p,2},
\]

and

\[
\left( \max_{\rho_{p,2}} \right)^{-(r+1)} \leq d_{\Omega_4}^2 \left( \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n} (y_n + B), 0 \right) \leq \left( \max_{\rho_{p,2}} \right)^{-r}. \tag{4.29}
\]

Since \( \Omega_4 \) is compact, we can choose a subsequence \( n_k \) such that

\[
\lim_{k \to \infty} \left( \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n_k} (y_{n_k} + B) \right) = y + B \neq 0
\]

and

\[
\left( \max_{\rho_{p,2}} \right)^{-(r+1)} \leq d_{\Omega_4}^2 (y + B, 0) \leq \left( \max_{\rho_{p,2}} \right)^{-r}. \tag{4.30}
\]

By (4.29) and (4.27) we have

\[
d_{\Omega_4}^2 \left( \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n_k} y_{n_k} + B, 0 \right) \geq \left( \min_{\rho_{p,2}} \rho'_{p,2} \right)^{l_n_k} \frac{d_{\Omega_4}^2 (y_n + B, 0)}{\max_{\rho_{p,1}} \rho_{p,1}} \frac{d_{\Omega_4}^2 (x_n + B, 0)}{\max_{\rho_{p,2}} \rho_{p,2}}.
\]

Now (4.30) and (4.19) together with our assumption that the limit in (4.18) is non-zero or \(+ \infty\) imply that

\[
\lim_{k \to \infty} \frac{d_{\Omega_4}^2 \left( \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n_k} y_{n_k} + B, 0 \right)}{d_{\Omega_4}^1 \left( \left( s_1^{(\alpha_0, \beta_0)} \right)^{l_n_k} x_{n_k} + B, 0 \right)} = + \infty.
\]

This and (4.30) imply that

\[
d_{\Omega_4}^1 \left( \left( s_1^{(\alpha_0, \beta_0)} \right)^{l_n_k} x_{n_k} + B, 0 \right) \to 0 \quad \text{as} \quad k \to \infty.
\]

Thus, we have a sequence of points

\[
\left\{ \left( \left( s_1^{(\alpha_0, \beta_0)} \right)^{l_n_k} x_{n_k} + B, \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n_k} y_{n_k} + B \right) \right\} \subset \Omega
\]

such that

\[
\left( \left( s_1^{(\alpha_0, \beta_0)} \right)^{l_n_k} x_{n_k} + B, \left( s_2^{(\alpha_0, \beta_0)} \right)^{l_n_k} y_{n_k} + B \right) \to (B y + B) \in \{0\} \times \Omega_4,
\]

with

\[
y + B \neq 0 \quad \text{and} \quad \left( \max_{\rho_{p,2}} \right)^{-(r+1)} \leq d_{\Omega_4}^2 (y + B, 0) \leq \left( \max_{\rho_{p,2}} \right)^{-r}.
\]

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Repeating this construction for the sequence of natural numbers \( r \to \infty \) we get a sequence of different points in \( \{ \{0\} \times \Omega_a^4 \} \cap \Omega \) tending to zero. This contradicts (4.17).

If the limit in (4.18) is zero, then we proceed analogously to the previous case changing the role of coordinates in \( \Omega_a^4 \). By Lemma 4.2 there exists an element \( s^{(\alpha_0',\beta_0')} \in \Sigma \) such that \( \min_{p \in S} \rho_{p,1}' > \max_{p \in S} \rho_{p,2} > 1 \). By Lemma 4.16 there exist norms \( \| \cdot \|_{p,j,i} \) in \( V_{i,j} \) such that, for every \( x \in V_{1,j} \),

\[
\rho_{p,j,1} \| x \|_{p,j,1} \geq \left\| s_1^{(\alpha_0',\beta_0')} x \right\|_{p,j,1} \geq \rho_{p,j,1}' \| x \|_{p,j,1},
\]

and for every \( y \in V_{2,j} \) we have,

\[
\left\| s_2^{(\alpha_0',\beta_0')} y \right\|_{p,j,2} \leq \rho_{p,j,2} \| y \|_{p,j,2}.
\]

Now we consider

\[
d_i^{\Omega_a^4}, \ i = 1, 2,
\]

defined by (4.5) with the use of \( \| \cdot \|_{p,i} \) defined above, and we prove the analogue of Lemma 4.22 saying that \( s_1^{(\alpha_0',\beta_0')} \) is expansive. This allows us to construct a sequence of points

\[
(y_r + B, B) \in \Omega_a^4 \times \{0\},
\]

with

\[
y_r + B \neq 0, \ (\max_{p \in S} \rho_{p,1})^{-(r+1)} \leq d_1^{\Omega_a^4} (y_r + B, 0) \leq (\max_{p \in S} \rho_{p,1})^{-r}.
\]

This again contradicts (4.17). \( \square \)

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