THE $b$-ADIC DIAPHONY OF DIGITAL SEQUENCES

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ABSTRACT. The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit cube. In this article we give a formula for the $b$-adic diaphony of digital $(0,s)$-sequences over $\mathbb{Z}_b$, $s = 1, \ldots, b$. This formula shows that for a fixed $s \in \{1, \ldots, b\}$, the $b$-adic diaphony has the same values for any digital $(0,s)$-sequence over $\mathbb{Z}_b$. For $t > 0$ we show upper bounds on the $b$-adic diaphony of digital $(t,s)$-sequences over $\mathbb{Z}_b$. We also consider the asymptotic behavior of the $b$-adic diaphony of these digital sequences.

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1. Introduction

The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the $s$-dimensional unit cube. This notion was introduced by Hellekalek and Leeb [11] for $b = 2$ and later generalized by Grozdanov and Stoitova [10] for general integers $b \geq 2$. The main difference to the classical diaphony is that the trigonometric functions are replaced by $b$-adic Walsh functions. Before we give the exact definition of the $b$-adic diaphony we recall the definition of Walsh functions.

Let $b \geq 2$ be an integer. For a nonnegative integer $k$ with base $b$ representation $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_1 b + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function $b\text{wal}_k : [0,1) \to \mathbb{C}$ by

$$b\text{wal}_k(x) := e^{2\pi i (x_1 \kappa_0 + \cdots + x_{a-1} \kappa_{a-1})/b},$$

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for \( x \in [0, 1) \) with base \( b \) representation \( x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots \) (unique in the sense that infinitely many of the \( x_i \) must be different from \( b - 1 \)).

For dimensions \( s \geq 2 \), \( x_1, \ldots, x_s \in [0, 1) \) and \( k_1, \ldots, k_s \in \mathbb{N}_0 \) we define \( b_{wal_{k_1, \ldots, k_s}} : [0, 1)^s \to \mathbb{C} \) by

\[
b_{wal_{k_1, \ldots, k_s}}(x_1, \ldots, x_s) := \prod_{j=1}^s b_{wal_{k_j}}(x_j).
\]

For vectors \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) and \( x = (x_1, \ldots, x_s) \in [0, 1)^s \) we write

\[
b_{wal_k}(x) := b_{wal_{k_1, \ldots, k_s}}(x_1, \ldots, x_s).
\]

Now we give the definition of the \( b \)-adic diaphony (see [10] or [11]).

**Definition 1.** Let \( b \geq 2 \) be an integer. The \( b \)-adic diaphony of the first \( N \) elements of a sequence \( \omega = (x_n)_{n \geq 0} \) in \([0, 1)^s\) is defined by

\[
F_{b,N}(\omega) := \left( \frac{1}{(1 + b)^s - 1} \sum_{k \in \mathbb{N}_0^s \setminus k \neq 0} r_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} b_{wal_k}(x_n) \right|^2 \right)^{1/2},
\]

where for \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \), \( r_b(k) := \prod_{j=1}^s r_b(k_j) \) and for \( k \in \mathbb{Z} \),

\[
r_b(k) := \begin{cases} 
1 & \text{if } k = 0, \\
b^{-2a} & \text{if } b^a \leq k < b^{a+1}, \text{ where } a \in \mathbb{N}_0.
\end{cases}
\]

Throughout this article we will write \( a(k) = a \), if \( a \) is the unique determined integer such that \( b^a \leq k < b^{a+1} \). If \( b = 2 \) we also speak of dyadic diaphony.

The \( b \)-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence \( \omega \) in the \( s \)-dimensional unit cube is uniformly distributed modulo one if and only if \( \lim_{N \to \infty} F_{b,N}(\omega) = 0 \). This was shown in [11] for the case \( b = 2 \) and in [10] for the general case. Further it is shown in [3] that the \( b \)-adic diaphony is – up to a factor depending on \( b \) and \( s \) – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space \( H_{wal,s,\gamma} \), which has been introduced in [5].

Throughout this paper let \( b \) be a prime and let \( \mathbb{Z}_b \) be the finite field of prime order \( b \). We consider the \( b \)-adic diaphony of a special class of sequences in \([0, 1)^s\), namely of so-called digital \((t,s)\)-sequences over \( \mathbb{Z}_b \). Hereby \( s \) is the dimension of the sequence and \( t \geq 0 \) is a quality parameter; lower values of \( t \) imply stronger equidistribution properties. Digital \((t,s)\)-sequences were introduced by Niederreiter [14, 15] in a more general setting and they provide at the moment the most efficient method to generate sequences with excellent distribution properties.
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We remark that a digital $(0, s)$-sequence over $\mathbb{Z}_b$ only exists if $s \in \{1, \ldots, b\}$. For higher dimensions $s \geq b + 1$ the concept of digital $(t, s)$-sequences over $\mathbb{Z}_b$ with $t > 0$ has to be stressed (see [14], [15] or [4]).

Before we give the definition of digital $(t, s)$-sequences over $\mathbb{Z}_b$ we introduce some notation: for a vector $c = (c_1, c_2, \ldots) \in \mathbb{Z}_b^\infty$ and for $m \in \mathbb{N}$ we denote the vector in $\mathbb{Z}_b^m$ consisting of the first $m$ components of $c$ by $c(m)$, i.e., $c(m) = (c_1, \ldots, c_m)$. Further for an $N \times N$ matrix $C$ over $\mathbb{Z}_b$ and for $m \in \mathbb{N}$ we denote by $C(m)$ the left upper $m \times m$ sub-matrix of $C$.

**Definition 2.** Let $b$ be a prime. For $s \in \mathbb{N}$ and $t \in \mathbb{N}_0$, choose $s \mathbb{N} \times \mathbb{N}$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$ with the following property: for every $m \in \mathbb{N}, m \geq t$ and every $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m - t$ the vectors

$$c_1^{(1)}(m), \ldots, c_{d_1}^{(1)}(m), \ldots, c_1^{(s)}(m), \ldots, c_{d_s}^{(s)}(m)$$

are linearly independent in $\mathbb{Z}_b^m$. Here $c_i^{(j)}$ is the $i$th row vector of the matrix $C_j$.

For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base $b$ representation of $n$. For $j \in \{1, \ldots, s\}$ multiply the vector $n = (n_0, n_1, \ldots)^\top$ by the matrix $C_j$,

$$C_j \cdot n = (x_n^j(1), x_n^j(2), \ldots)^\top \in \mathbb{Z}_b^\infty,$$

and set

$$x_n^{(j)} := \frac{x_n^j(1)}{b} + \frac{x_n^j(2)}{b^2} + \cdots$$

Finally, set

$$x_n := (x_n^{(1)}, \ldots, x_n^{(s)}).$$

Every sequence $(x_n)_{n \geq 0}$ constructed this way is called digital $(t, s)$-sequence over $\mathbb{Z}_b$. The matrices $C_1, \ldots, C_s$ are called the generator matrices of the sequence.

**Remark 3.** Let $(x_n)_{n \geq 0}$ be a digital $(t, s)$-sequence over $\mathbb{Z}_b$. For $\emptyset \neq u \subseteq \{1, \ldots, s\}$ we define $x_n^{(u)}$ as the projection of $x_n$ onto the coordinates in $u$. The sequence $(x_n^{(u)})_{n \geq 0}$ is now a digital $(t, |u|)$-sequence over $\mathbb{Z}_b$.

To guarantee that the points $x_n$ belong to $[0, 1]^s$ (and not just to $[0, 1]^s$) and also for the analysis of the sequence we need the condition that for each $n \geq 0$ and $1 \leq j \leq s$, we have $x_n^i(i) = 0$ for infinitely many $i$. This condition is always satisfied if we assume that for each $1 \leq j \leq s$ and $r \geq 0$ we have $c_{i,r}^{(j)} = 0$ for all sufficiently large $i$, where $c_{i,r}^{(j)}$ are the entries of the matrix $C_j$. Throughout this article we assume that the generator matrices fulfil this condition (see [15] p. 72] where this condition is called (S6)). More information about $(t, s)$-sequences can be found in the books [15] and [4].
The classical diaphony $F_N$ (see [17] or [7] Definition 1.29 or [13] Exercise 5.27, p. 162) of the first $N$ elements of a sequence $\omega = (x_n)_{n \geq 0}$ in $[0, 1)^s$ is given by

$$F_N(\omega) = \left( \sum_{k \in \mathbb{Z} \setminus k \neq 0} \frac{1}{\rho(k)^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \langle k, x_n \rangle} \right|^2 \right)^{1/2},$$

where for $k = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ it is $\rho(k) = \prod_{i=1}^s \max(1, |k_i|)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^s$. It is well known that the diaphony is a quantitative measure for the irregularity of distribution of the first $N$ points of a sequence: A sequence $\omega$ is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_N(\omega) = 0$.

For the classical diaphony it was proved by Faure [8, Theorem 4] that

$$\left( NF_N(\omega) \right)^2 = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \chi_{\delta j}^{\delta j - 1} \left( \frac{N}{b^j} \right), \quad (1)$$

where $\omega$ is a digital $(0, 1)$-sequence over $\mathbb{Z}_b$ whose generator matrix $C$ is a non-singular upper triangular matrix. Faure (and we shall do so as well) called these sequences NUT-sequences. Here $\chi_{\delta j}^{\delta j - 1}$ are certain functions depending on the generating matrix $C$. For an exact definition of $\chi_{\delta j}^{\delta j - 1}$ see [8].

The aim of this paper is to prove a similar formula for the $b$-adic diaphony of digital $(0, s)$-sequences over $\mathbb{Z}_b$ for $s \in \{1, \ldots, b\}$. This formula shows that for fixed $s$ the $b$-adic diaphony is invariant for digital $(0, s)$-sequences over $\mathbb{Z}_b$. For digital $(t, s)$-sequences over $\mathbb{Z}_b$, $t > 0$, we give upper bounds for the $b$-adic diaphony. We will also consider the asymptotic behavior. For $b = 2$ these results were already shown by Pillichshammer [16] and Kritzer and Pillichshammer in [12].

2. Results

We will show a formula for the $b$-adic diaphony of digital $(0, s)$-sequences over $\mathbb{Z}_b$ that has the same structure as (1). But instead of the functions $\chi_{\delta j}^{\delta j - 1}$ we will get a function $\psi_b$, which does not depend on the generating matrices. In Section 3 we will show some similarities between $\psi_b$ and $\chi_{\delta j}^{\delta j - 1}$. 

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**Definition 4.** Let \( \beta \) be an integer in \( \{1, \ldots, b-1\} \). For \( x \in \left[ \frac{j}{b}, \frac{j+1}{b} \right) \), \( j \in \{0, \ldots, b-1\} \) we set
\[
\psi_b^\beta(x) := \frac{b^2(b^2-1)}{12} \left| \frac{1}{b} \left( \frac{z^j}{z^\beta - 1} + z^j \left( x - \frac{j}{b} \right) \right) \right|^2,
\]
where \( z^\beta = e^{\frac{2\pi i}{b^\beta}} = \omega_b^1(\beta) \); then the function is extended to the reals by periodicity. The function \( \psi_b \) is now defined as the mean of the functions \( \psi_b^\beta \):
\[
\psi_b(x) := \frac{1}{b-1} \sum_{\beta=1}^{b-1} \psi_b^\beta(x).
\]

**Remark 5.** For \( b = 2 \) we have \( \psi_2(x) = \| \cdot \|^2 \), where \( \| \cdot \| \) denotes the distance to the nearest integer function, i.e., \( \|x\| := \min(x - \lfloor x \rfloor, 1 - (x - \lfloor x \rfloor)) \).

**Theorem 6.** Let \( b \) be a prime.

1. Let \( \omega \) be a digital \((0, s)\)-sequence over \( \mathbb{Z}_b \). Then for any \( N \geq 1 \) we have
\[
\left( NF_{b,N}(\omega) \right)^2 = \sum_{u=1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) P_s(u),
\]
where \( P_s(u) \) is a polynomial of degree \( s-1 \) in \( u \) defined by
\[
P_s(u) := \frac{1}{(b+1)^s-1} \frac{12}{b^2(b+1)} \left( \sum_{v=1}^{s} \binom{s}{v} (b-1)^v \sum_{w=0}^{\infty} \left( \frac{w+u+v-1}{w+u} \right) b^{-w} \right.
\]
\[
+ \sum_{v=1}^{s} \binom{s}{v} \sum_{l=1}^{v} b^{2l} \left( \frac{u-l+v-1}{u-l} \right) \sum_{j=0}^{v-l} \binom{v}{j} (-1)^j b^{v-l-j} \right).
\]

2. Let \( \omega \) be a digital \((t, s)\)-sequence over \( \mathbb{Z}_b \). Then for any \( N \geq 1 \) we have
\[
\left( NF_{b,N}(\omega) \right)^2 \leq \frac{1}{(b+1)^s-1} \sum_{v=1}^{s} \binom{s}{v} b^{2t+3v}
\]
\[
\times \left( \frac{b+1}{3b} + \frac{12}{b^2(b+1)} \sum_{u=t+v+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{v-1} \right).
\]

The proof of this Theorem will be given in Section 3.

**Remark 7.** This theorem shows that the \( b \)-adic diaphony is invariant for all digital \((0, s)\)-sequences over \( \mathbb{Z}_b \).
Remark 8. \( P_s(u) \) is a polynomial in \( u \) with degree \( s - 1 \) whose coefficients depend on \( b \) and \( s \). The leading coefficient of \( P_s(u) \) equals
\[
\frac{12(b^2 - 1)^{s-1}}{(s-1)!b((b+1)^s - 1)}.
\]
The first polynomials are:
\[
P_1(u) = \frac{12}{b^2},
\]
\[
P_2(u) = -\frac{12(b-2)(b+1)}{b^2(b+2)} + \frac{12(b-1)(b+1)u}{b^2(b+2)},
\]
\[
P_3(u) = \frac{12(b+1)^2(b^2 - 3b + 3)}{b^2(b^2 + 3b + 3)} - \frac{6(b-1)(b+1)^2(3b-5)u}{b^2(b^2 + 3b + 3)}
\]
\[
+ \frac{6(b-1)^2(b+1)^2u^2}{b^2(b^2 + 3b + 3)}.
\]
Hence for a digital \((0,1)\)-sequence over \( \mathbb{Z}_b \) and any \( N \geq 1 \) we have
\[
\left( NF_{b,N}(\omega) \right)^2 = \frac{12}{b^2} \sum_{u=1}^{\infty} \psi_b \left( \frac{N}{b^u} \right).
\]
(2)

We want to point out the similar structure of (1) and (2).

We now consider the asymptotic behavior of the \( b \)-adic diaphony of digital \((t,s)\)-sequences.

Corollary 9. The \( b \)-adic diaphony of a digital \((0,s)\)-sequence \( \omega \) over \( \mathbb{Z}_b \) is of order
\[
F_{b,N}(\omega) = \mathcal{O}\left( \frac{(\log N)^{s/2}}{N} \right).
\]
In particular we have
\[
\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\frac{2 \cdot 3^{s-2}}{(\log 2)^s(s-1)!(3^s - 1)}} \quad \text{if } b = 2,
\]
(3)

and
\[
\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\frac{(b^2 - 1)^s}{4(\log b)^s(s-1)!(b+1)^s - 1}}} \quad \text{if } b \text{ is odd.}
\]
(4)
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For $s = 1$ we have equality in (3) and (4), i.e.,

$$\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{1}{3 \log 2}}, \quad \text{if } b = 2,$$

and

$$\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{(b^2 - 1)}{4b \log b}}, \quad \text{if } b \text{ is odd}$$

if $\omega$ is a digital $(0,1)$-sequence over $\mathbb{Z}_b$.

The proof of this Corollary will be given in Section 3.

**Corollary 10.** Let $\omega$ be a digital $(t,s)$-sequence over $\mathbb{Z}_b$ and let $b^{m-1} < N \leq b^m$ with $m > t + s$. Then we have

$$\left( NF_{b,N}(\omega) \right)^2 \leq \frac{1}{(b + 1)^s - 1} \sum_{v=1}^{s} \binom{s}{v} b^{2t+3v} \left( \frac{b+1}{3b} + \frac{b^2 - 1}{3b^2} (m - t - 1)^v + (b - 1)(m - t - 1)^{v-1} c_v \right),$$

where

$$c_v = \sum_{u=1}^{\infty} \frac{(u+1)^{v-1}}{b^{2u}} < \infty.$$  

In particular,

$$F_{b,N}(\omega) = O \left( \frac{b^t (\log N)^{s/2}}{N} \right)$$

and we have

$$\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \frac{1}{\sqrt{(b + 1)^s - 1}} b^t \left( \frac{b^3}{\log b} \right)^{s/2}.$$

The proof of this Corollary will be given in Section 3.

### 3. Proofs

We will show some useful facts about $\psi_b$: For $b = 2$ and $b = 3$ the functions $\chi_{b,j-1}$ and $\psi_b$ are the same. Hence we have for any digital $(0,1)$-NUT-sequence $\omega$

$$F_{2,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega) \quad \text{and} \quad F_{3,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega), \quad (5)$$
i.e., the classical diaphony and the \( b \)-adic diaphony are for \( b = 2, 3 \) up to the constant \( \sqrt{\frac{2}{\pi}} \) the same. For \( b = 5 \) the functions \( \chi_b^{\delta_j-1} \) and \( \psi_b \) are no longer the same and there is no relation like (3), i.e.,

\[
F_{5,1}(\omega)/F_1(\omega) \neq F_{5,2}(\omega)/F_2(\omega)
\]

for all digital \((0,1)\)-NUT-sequences \( \omega \). This can be easily checked by calculation of \( \chi_b^{\delta_j-1} \) and \( \psi_b \). Nonetheless, the functions still have a similar structure. The next lemma shows some properties of the function \( \psi_b \). Compare this with properties of \( \chi_b^{\delta_j-1} \) in \cite[Propriété 3.3, Propriété 3.5 (ii)]{1}.

**Lemma 11** (Properties of \( \psi_b \)). We have:

1. \( \psi_b(x) = \frac{b^2(b^2-1)}{12} x^2 \) for \( x \in [0, \frac{1}{b}) \),
2. \( \psi_b \) is continuous,
3. on intervals of the form \([\frac{k}{b}, \frac{k+1}{b})\) the function \( \psi_b \) is a translation of the parabola \( \frac{b^2(b^2-1)}{12} x^2 \).

**Proof.** These properties follow easily from the definition of \( \psi_b \). \( \square \)

**Lemma 12.** The function \( \psi_b \) is bounded. In particular for all \( x \in \mathbb{R} \) we have

\[
\psi_b(x) \leq \frac{(b+1)(b^2-1)}{36}.
\]

**Proof.** Since \( \psi^\beta \) is on \([\frac{j}{b}, \frac{j+1}{b})\), \( j \in \{0, \ldots, b-1\} \), a translation of the parabola \( \frac{b^2(b^2-1)}{12} x^2 \), it attains its maximum in \( \frac{j}{b} \), \( j \in \{0, \ldots, b-1\} \). So there exists a \( j \in \{0, \ldots, b-1\} \) such that

\[
\psi^\beta_b(x) \leq \psi^\beta_b \left( \frac{j}{b} \right) = \frac{b^2(b^2-1)}{12} \left| 1 - \frac{e^{2\pi i \beta j}}{b} \frac{e^{2\pi i \beta j}}{b} - 1 \right|^2 
\leq \frac{b^2 - 1}{12} \left| \frac{e^{\pi i \beta}}{b} \right|^2 \left| 2i \sin \left( \frac{\pi \beta}{b} \right) \right|^2 
= \frac{b^2 - 1}{12} \left( \frac{\pi \beta}{b} \right)^2.
\]

So we get

\[
\psi_b(x) = \frac{1}{b-1} \sum_{\beta=1}^{b-1} \psi^\beta_b(x) \leq \frac{b+1}{12} \sum_{\beta=1}^{b-1} \frac{1}{\sin^2 \left( \frac{\pi \beta}{b} \right)} = \frac{(b+1)(b^2-1)}{36},
\]
where we used the fact that
\[
\sum_{\beta=1}^{b-1} \frac{1}{\sin^2 \left( \frac{\pi \beta}{b} \right)} = \frac{b^2 - 1}{3},
\]
see [5, Appendix C] or [4, Corollary A.23].

**Lemma 13.**
1. \( \psi_b \left( \frac{k}{b} \right) = \frac{(b+1)k(b-k)}{12} \) for \( k \in \{0, \ldots, b-1\} \).
2. \( \psi_b \) is symmetric, i.e., \( \psi_b(x) = \psi_b(1-x) \) for all \( x \in [0, 1] \).
3. \( \psi_b \left( \frac{1}{2} \right) = \frac{b(b^2-1)}{48} \) if \( b \) is odd.

**Proof.**
1. Let \( k \in \{0, \ldots, b-1\} \). Then we have
\[
\psi_b \left( \frac{k}{b} \right) = \frac{1}{b-1} \sum_{\beta=1}^{b-1} \frac{b^2(b^2 - 1)}{12} \left| \frac{1}{b} e^{\frac{2\pi i}{b} \beta k} - 1 \right|^2
\]
\[
= \frac{b+1}{24} \sum_{\beta=1}^{b-1} \frac{1}{\sin^2 \left( \frac{\pi \beta}{b} \right)} \cos \left( \frac{2\pi \beta k}{b} \right) + \frac{b+1}{24} \frac{1}{\sin^2 \left( \frac{\pi \beta}{b} \right)} \sum_{\beta=1}^{b-1} \frac{e^{\frac{2\pi i}{b} \beta k}}{\sin^2 \left( \frac{\pi \beta}{b} \right)}
\]
\[
= \frac{(b+1)(b^2-1)}{24} - \frac{b+1}{24} \frac{1}{3} + \frac{2k(k-b)}{24}
\]
\[
= \frac{(b+1)k(b-k)}{12},
\]
where we used the fact that
\[
\sum_{\beta=1}^{b-1} \frac{1}{\sin^2 \left( \frac{\pi \beta}{b} \right)} = \frac{b^2 - 1}{3} \quad \text{and} \quad \sum_{\beta=1}^{b-1} \frac{e^{\frac{2\pi i}{b} \beta}}{\sin^2 \left( \frac{\beta \pi}{b} \right)} = \frac{b^2 - 1}{3} + 2|k||\lfloor k \rfloor - b|
\]
for \( k \in \{-b, \ldots, b\} \), see [5, Appendix C] or [4, Corollary A.23].

2. Since \( \psi_b \left( \frac{k}{b} \right) = \psi_b \left( \frac{b-k}{b} \right) \) for \( k \in \{0, \ldots, b-1\} \) and \( \psi_b \) consists of translations of the same parabola (see Lemma 11), \( \psi_b \) must be symmetric.

3. Let \( b = 2p + 1 \). Then we have \( \frac{1}{2} \in \left[ \frac{b}{b}, \frac{p+1}{b} \right) \). On this interval \( \psi_b \) is of the form
\[
\psi_b(x) = \frac{b^2(b^2 - 1)}{12} x^2 + ax + c.
\]
We determine now the constants $a$ and $c$. Since 

$$\psi_b \left( \frac{p}{b} \right) = \psi_b \left( \frac{p+1}{b} \right)$$

we have 

$$\psi'_b \left( \frac{1}{2} \right) = 0.$$ 

From this we obtain 

$$a = \frac{b^2 (b^2 - 1)}{12}.$$ 

From 

$$\psi_b \left( \frac{p}{b} \right) = \frac{(b+1)(b^2 - 1)}{48}$$

we obtain 

$$c = \frac{b(b+1)(b^2 - 1)}{48}.$$ 

Now we can compute 

$$\psi_b \left( \frac{1}{2} \right) = \frac{b^2 (b^2 - 1)}{48} - \frac{b^2 (b^2 - 1)}{24} + \frac{b(b+1)(b^2 - 1)}{48} = \frac{b(b^2 - 1)}{48}.$$ 

\[\square\]

**Lemma 14.** Let the nonnegative integer $U$ have $b$-adic expansion 

$$U = U_0 + U_1 b + \cdots + U_{m-1} b^{m-1}.$$ 

For any nonnegative integer $n \leq U - 1$ let 

$$n = n_0 + n_1 b + \cdots + n_{m-1} b^{m-1}$$

be the $b$-adic representation of $n$. For $0 \leq p \leq m - 1$ let 

$$U(p) := U_0 + \cdots + U_p b^p.$$ 

Let 

$$\beta_0, \beta_1, \ldots, \beta_{m-1}$$

be arbitrary elements of $\mathbb{Z}_b$, not all zero. Then 

$$\sum_{n=0}^{U-1} e^{\frac{2\pi i}{b}(\beta_0 n_0 + \cdots + \beta_{m-1} n_{m-1})} = e^{\frac{2\pi i}{b}(\beta_w U_{w+1} + \cdots + \beta_{m-1} U_{m-1})} b^{w+1} \theta,$$

where 

$$\theta := \frac{1}{b} \frac{z^j - 1}{z - 1} + z^j \left( \frac{U(w)}{b^{w+1}} - \frac{j}{b} \right)$$

if 

$$\frac{U(w)}{b^{w+1}} \in \left[ \frac{j}{b}, \frac{j+1}{b} \right), \quad j = 0, \ldots, b - 1, \quad z := e^{\frac{2\pi i}{b} \beta_w}$$

and $w$ is minimal such that $\beta_w \neq 0$. 

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Proof. From splitting the sum we obtain

\[
\begin{align*}
&\sum_{n=0}^{U-1} e^{\frac{2\pi i}{b} (\beta_0 n_0 + \cdots + \beta_{m-1} n_{m-1})} \\
&= b^{w+1} (U_{w+1} + \cdots + U_{m-1} b^{m-w-2})^{-1} \\
&\quad + \sum_{n=b^{w+1}(U_{w+1} + \cdots + U_{m-1} b^{m-w-2})}^{U-1} e^{\frac{2\pi i}{b} (\beta_0 n_0 + \cdots + \beta_{m-1} n_{m-1})} \\
&= 0 + \sum_{n=0}^{U(w)-1} e^{\frac{2\pi i}{b} (\beta_w n_w + \beta_{w+1} U_{w+1} + \cdots + \beta_{m-1} U_{m-1})} \\
&= e^{\frac{2\pi i}{b} (\beta_{w+1} U_{w+1} + \cdots + \beta_{m-1} U_{m-1})} \sum_{n=0}^{U(w)-1} e^{\frac{2\pi i}{b} \beta_w n_w}.
\end{align*}
\]

We consider now the last sum. If $U_w = j$ for $j \in \{0, 1, \ldots, b - 1\}$, then we have:

\[
\begin{align*}
\sum_{n=0}^{U(w)-1} e^{\frac{2\pi i}{b} \beta_w n_w} &= b^{w-1} + \sum_{n=b^w}^{2b^w-1} e^{\frac{2\pi i}{b} \beta_w} + \sum_{n=0}^{2b^w-1} e^{\frac{2\pi i}{b} \beta_w} + \cdots \\
&\quad + \sum_{n=(j-1)b^w}^{jb^w-1} e^{\frac{2\pi i}{b} \beta_w (j-1)} + \sum_{n=jb^w}^{U(w)-1} e^{\frac{2\pi i}{b} \beta_w j} \\
&= b^w + b^w e^{\frac{2\pi i}{b} \beta_w} + \cdots + b^w e^{\frac{2\pi i}{b} \beta_w (j-1)} + (U(w) - jb^w) e^{\frac{2\pi i}{b} \beta_w j} \\
&= b^{w+1} \left( \frac{1}{b} + \frac{z}{b} + \cdots + \frac{z^{j-1}}{b} - j \frac{z^j}{b} + z^j \frac{U(w)}{b^{w+1}} \right) \\
&= b^{w+1} \left( \frac{1}{b} + \frac{z^j - 1}{z - 1} + z^j \left( \frac{U(w)}{b^{w+1}} - \frac{j}{b} \right) \right).
\end{align*}
\]

Since $U_w = j$ means $U(w) \in [jb^w, (j + 1)b^w)$ and $\frac{U(w)}{b^{w+1}} \in [\frac{j}{b}, \frac{j+1}{b})$ the result follows.

\[\square\]

Proof of Theorem 6. Let $\omega$ be a digital $(t, s)$-sequence over $\mathbb{Z}_b$, $t \geq 0$. For a point $x_n$ of $\omega$ and for $\emptyset \neq u \subseteq \{1, \ldots, s\}$, we define $x_n^{(u)}$ as the projection
of $x_n$ onto the coordinates in $u$. We have

$$\left( NF_{b,N}(\omega) \right)^2$$

$$= \frac{1}{(b+1)^s-1} \sum_{k \in \mathbb{N}_0 \setminus k \neq 0} r_b(k) \left| \sum_{n=0}^{N-1} b \text{wal}_k(x_n) \right|^2$$

$$= \frac{1}{(b+1)^s-1} \sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \sum_{k \in \mathbb{N}_0 \setminus k \neq 0} r_b(k) \left| \sum_{n=0}^{N-1} b \text{wal}_k(x_n^{(u)}) \right|^2$$

$$= \frac{1}{(b+1)^s-1} \sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \sum_{u \subseteq \{w_1,\ldots,w_{|u|}\}} \cdots k_{w_1} = 1$$

$$\cdots \sum_{k_{w_{|u|}} = 1} \left( \prod_{j \in u} \frac{1}{b^{2a(k_j)}} \right) \left| \sum_{n=0}^{N-1} b \text{wal}_{(k_{w_1},\ldots,k_{w_{|u|}})}(x_n^{(u)}) \right|^2.$$  

Let now $\emptyset \neq u = \{w_1,\ldots,w_{|u|}\} \subseteq \{1,\ldots,s\}$ be fixed. We have to study

$$\Sigma(u) := \sum_{k_{w_1} = 1}^{\infty} \cdots \sum_{k_{w_{|u|}} = 1}^{\infty} \left( \prod_{j \in u} \frac{1}{b^{2a(k_j)}} \right) \left| \sum_{n=0}^{N-1} b \text{wal}_{(k_{w_1},\ldots,k_{w_{|u|}})}(x_n^{(u)}) \right|^2.$$  

For the sake of simplicity we assume in the following $u = \{1,\ldots,\sigma\}$, $1 \leq \sigma \leq s$. The other cases are dealt with a similar fashion. We have

$$\Sigma(\{1,\ldots,\sigma\}) = \sum_{k_1 = 1}^{\infty} \cdots \sum_{k_{\sigma} = 1}^{\infty} \left( \prod_{j=1}^{\sigma} \frac{1}{b^{2a(k_j)}} \right) \left| \sum_{n=0}^{N-1} b \text{wal}_k(x_n^{\{1,\ldots,\sigma\}}) \right|^2,$$

where $k_{\sigma} := (k_1,\ldots,k_{\sigma}).$

For $1 \leq j \leq \sigma$, let $b^{a_j} \leq k_j < b^{a_j+1}$, then $k_j = \kappa_{0}^{(j)} + \kappa_{1}^{(j)} b + \cdots + \kappa_{a_j}^{(j)} b^{a_j}$ with $\kappa_{0}^{(j)} \in \{0,1,\ldots,b-1\}$, $0 \leq v < a_j$, and $\kappa_{a_j}^{(j)} \in \{1,\ldots,b-1\}$.

Let $c_j^{(i)}$ be the $i$-th row vector of the generator matrix $C_j$, $1 \leq j \leq \sigma$. Since the $i$-th digit $x_n^{(j)}(i)$ of $x_n^{(j)}$ is given by $\langle c_j^{(i)}, n \rangle$, we have

$$\sum_{n=0}^{N-1} b \text{wal}_k(x_n^{\{1,\ldots,\sigma\}}) = \sum_{n=0}^{N-1} e^{\frac{2\pi i}{b} \sum_{j=1}^{\sigma} (\kappa_{0}^{(j)} \langle c_1^{(i)}, n \rangle + \cdots + \kappa_{a_j}^{(j)} \langle c_{a_j}^{(i)}, n \rangle)}$$

$$= \sum_{n=0}^{N-1} e^{\frac{2\pi i}{b} \sum_{j=1}^{\sigma} (\kappa_{0}^{(j)} c_1^{(i)} + \cdots + \kappa_{a_j}^{(j)} c_{a_j}^{(i)}, n)}.$$
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Let

$$C_j = \left( c_{v,w}^{(j)} \right)_{v,w \geq 1}.$$  

Define

$$u(k_\sigma) := \min \left\{ l \geq 1 : \sum_{j=1}^{\sigma} \left( \kappa_0^{(j)} c_{1,l}^{(j)} + \cdots + \kappa_{a_j} c_{a_j+1,l}^{(j)} \right) \neq 0 \right\}$$

and

$$\beta_{k_\sigma} = (\beta_{k_\sigma 0}, \beta_{k_\sigma 1}, \ldots)^\top := \sum_{j=1}^{\sigma} \left( \kappa_0^{(j)} c_{1}^{(j)} + \cdots + \kappa_{a_j} c_{a_j+1}^{(j)} \right).$$

Since $C_1, \ldots, C_s$ generate a digital $(t, s)$-sequence over $\mathbb{Z}_b$, it is easy to verify $u(k_\sigma) \leq \sum_{j=1}^{\sigma} a_j + \sigma + t =: R_\sigma + \sigma + t$. Indeed, let

$$C(a_1, \ldots, a_\sigma) := \begin{pmatrix} c_{1,1}^{(1)} & \cdots & c_{1,1}^{(a_1+1,1)} & \cdots & c_{1,1}^{(\sigma)} & \cdots & c_{1,1}^{(a_\sigma+1,1)} \\ c_{1,2}^{(1)} & \cdots & c_{1,2}^{(a_1+1,2)} & \cdots & c_{1,2}^{(\sigma)} & \cdots & c_{1,2}^{(a_\sigma+1,2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{1,R_\sigma+\sigma+t}^{(1)} & \cdots & c_{1,R_\sigma+\sigma+t}^{(a_1+1,R_\sigma+\sigma+t)} & \cdots & c_{1,R_\sigma+\sigma+t}^{(\sigma)} & \cdots & c_{1,R_\sigma+\sigma+t}^{(a_\sigma+1,R_\sigma+\sigma+t)} \end{pmatrix}.$$  

Note that $C = C(a_1, \ldots, a_\sigma)$ is an $(R_\sigma + \sigma + t) \times (R_\sigma + \sigma)$ matrix. Since $C_1, \ldots, C_s$ generate a digital $(t, s)$-sequence over $\mathbb{Z}_b$, it follows that $C(a_1, \ldots, a_\sigma)$ has rank $R_\sigma + \sigma$. If however, $u(k_\sigma) > R_\sigma + \sigma + t$, we would have

$$C \cdot \begin{pmatrix} \kappa_0^{(1)} \\ \vdots \\ \kappa_{a_1-1}^{(1)} \\ \kappa_{a_1}^{(1)} \\ \vdots \\ \kappa_{a_\sigma-1}^{(1)} \\ \kappa_{a_\sigma}^{(1)} \end{pmatrix} = (0, \ldots, 0)^\top,$$

which would lead to a contradiction since the matrix $C(a_1, \ldots, a_\sigma)$ has full rank and is multiplied by a nonzero vector in $\mathbb{R}$.

Let $N = N_0 + N_1 b + \cdots + N_{m-1} b^{m-1}$. If $u(k_\sigma) \leq m$ we obtain from Lemma [14]

$$\left| \sum_{n=0}^{N-1} \text{walk}_{k_\sigma} (x_n^{\{1, \ldots, \sigma\}}) \right|^2 = \left| \sum_{n=0}^{N-1} e^{2\pi i \langle \beta_{k_\sigma} n \rangle} \right|^2 = \left| \sum_{n=0}^{N-1} e^{2\pi i (n_0 \beta_{k_\sigma 0} + \cdots + n_{m-1} \beta_{k_\sigma,m-1})} \right|^2 = b^{2u(k_\sigma)} \left| \frac{1}{b} \frac{z^j - 1}{z - 1} + z^j \left( \frac{N(u(k_\sigma) - 1)}{b^{u(k_\sigma)}} - j \right) \right|^2,$$

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Since $\psi_{u_{\mathbf{\sigma}}}$, we have $\sum_{n=0}^{\infty} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma))$. Therefore, we have

\[
\left| \sum_{n=0}^{N-1} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma)) \right|^{2} = b^{2}u(\mathbf{\sigma}) \frac{12}{b^{2}(b^{2}-1)} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N(u(\mathbf{\sigma})-1)}{b^{u}(\mathbf{\sigma})} \right).
\]

Since $\psi_{b}^{\beta_{\mathbf{\sigma},u(\mathbf{\sigma})-1}}$ is 1-periodic and $\sum_{n=0}^{N-1} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma))$, we get

\[
\left| \sum_{n=0}^{N-1} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma)) \right|^{2} = \left| \sum_{n=0}^{N-1} e^{\frac{2 \pi i}{b^{u}(\mathbf{\sigma})}} \right|^{2} = N^{2} = b^{2}u(\mathbf{\sigma}) \frac{12}{b^{2}(b^{2}-1)} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N}{b^{u}(\mathbf{\sigma})} \right).
\]

If $u(k) > m$ we have

\[
\left| \sum_{n=0}^{N-1} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma)) \right|^{2} = \prod_{k_{1}=1}^{\infty} \cdots \prod_{k_{\sigma}=1}^{\infty} \left( \prod_{j=1}^{\sigma} \frac{1}{b^{2a_{\sigma}(k_{j})}} \right) b^{2u(\mathbf{\sigma})} \frac{12}{b^{2}(b^{2}-1)} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N}{b^{u}(\mathbf{\sigma})} \right).
\]

because $\sum_{n=0}^{N-1} b \omega_{\mathbf{\sigma}}(x_{n}(1,\ldots,\sigma)) \in [0, \frac{1}{b^{u}(\mathbf{\sigma})}]$. Therefore, we have

\[
\Sigma(\{1,\ldots,\sigma\}) = \sum_{k_{1}=1}^{\infty} \cdots \sum_{k_{\sigma}=1}^{\infty} \left( \prod_{j=1}^{\sigma} \frac{1}{b^{2a_{\sigma}(k_{j})}} \right) b^{2u(\mathbf{\sigma})} \frac{12}{b^{2}(b^{2}-1)} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N}{b^{u}(\mathbf{\sigma})} \right).
\]

\[
= \frac{12}{b^{2}(b^{2}-1)} \sum_{a_{\sigma}=0}^{\infty} \sum_{b_{\sigma}=0}^{\infty} \left( \prod_{j=1}^{\sigma} \frac{1}{b^{2a_{\sigma}(k_{j})}} \right) \sum_{k_{1}=b^{u_{1}}}^{b^{u_{1}}-1} \cdots \sum_{k_{\sigma}=b^{u_{\sigma}}}^{b^{u_{\sigma}}-1} b^{2u(\mathbf{\sigma})} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N}{b^{u}(\mathbf{\sigma})} \right).
\]

\[
= \frac{12}{b^{2}(b^{2}-1)} \sum_{a_{\sigma}=0}^{\infty} \sum_{b_{\sigma}=0}^{\infty} \frac{1}{b^{2R_{\sigma}}} \sum_{u_{1}=1}^{R_{\sigma}+\sigma+t} \cdots \sum_{u_{\beta}=1}^{b^{u_{\beta}}-1} b^{2u} \psi_{\mathbf{\sigma},u(\mathbf{\sigma})-1} \left( \frac{N}{b^{u}} \right) \sum_{k_{1}=b^{u_{1}}}^{b^{u_{1}}-1} \cdots \sum_{k_{\sigma}=b^{u_{\sigma}}}^{b^{u_{\sigma}}-1} 1.
\]
We need to evaluate the sum

\[
\sum_{k_1=b^1}^{b^{a+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1} 1
\]

for \(1 \leq u \leq R_\sigma + \sigma + t\) and \(\beta \in \{1, \ldots, b-1\}\). This is the number of digits

\[
\kappa_0^{(1)}, \ldots, \kappa_{a_1-1}^{(1)}, \theta_1, \ldots, \kappa_0^{(\sigma)}, \ldots, \kappa_{a_\sigma-1}^{(\sigma)}, \quad \theta_\sigma \in \{0, \ldots, b-1\}, \quad \theta_1 \neq 0, \ldots, \theta_\sigma \neq 0,
\]

such that

\[
C(a_1, \ldots, a_\sigma) \begin{pmatrix}
\kappa_0^{(1)} \\
\vdots \\
\kappa_{a_1-1}^{(1)} \theta_1 \\
\vdots \\
\kappa_0^{(\sigma)} \\
\vdots \\
\kappa_{a_\sigma-1}^{(\sigma)} \theta_\sigma
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\beta \\
x_{u+1} \\
\vdots \\
x_{R_\sigma+\sigma+t}
\end{pmatrix}
\]

for arbitrary \(x_{u+1}, \ldots, x_{R_\sigma+\sigma+t} \in \mathbb{Z}_b\).

Now we must distinguish between digital \((0, s)\)-sequences and digital \((t, s)\)-sequences, \(t > 0\). In the case of digital \((0, s)\)-sequences we can give the exact number of solutions of the required form of the above system. In the case of digital \((t, s)\)-sequences, \(t > 0\), we will use an upper bound.

First we deal with digital \((0, s)\)-sequences over \(\mathbb{Z}_b\). Note that since \(C_1, \ldots, C_s\) generate a digital \((0, s)\)-sequence it follows that \(C(a_1, \ldots, a_\sigma)\) is regular. So for any choice of the \(x_{u+1}, \ldots, x_{R_\sigma+\sigma}\) there exists one digit vector such that (7) is fulfilled. We only count those that have nonzero entries for \(\theta_1, \ldots, \theta_\sigma\).

We consider two cases:

(i) Assume that \(u = R_\sigma + l, l \in \{1, \ldots, \sigma\}\). Then we have \(b^{\sigma-l}\) choices for \(x_{u+1}, \ldots, x_{R_\sigma+\sigma}\). Choose \(j, j \leq \sigma - l\), of the \(\theta_1, \ldots, \theta_\sigma\) to be zero.

Now we construct a certain subsystem of (7) in the following way: Delete the column

\[
\begin{pmatrix}
c_{a_i+1,1}^{(i)} \\
c_{a_i+1,2}^{(i)} \\
\vdots \\
c_{a_i+1,R_\sigma+\sigma}^{(i)}
\end{pmatrix}^T
\]

from \(C(a_1, \ldots, a_\sigma)\) if \(\theta_i = 0\)
by choice and delete the last \(j\) lines. From the digit vector we delete those \(\theta_i\) we have chosen to be zero and from the right-hand-side vector we delete the last \(j\) entries. For arbitrary 
\[
x_{u+1}, \ldots, x_{R_\sigma + \sigma - j}
\]
there exists one solution of the corresponding subsystem. The remaining \(x_{R_\sigma + \sigma - j + 1}, \ldots, x_{R_\sigma + \sigma}\) are chosen properly such that the solution of the subsystem becomes a solution of the whole system. Hence there are \(b^{\sigma - l - j}\) solutions with \(j\) of the \(\theta_1, \ldots, \theta_\sigma\) chosen to be zero. If \(j > \sigma - l\), then there is no solution. Hence the number of solutions such that \(\theta_1 \neq 0, \ldots, \theta_\sigma \neq 0\) equals (using the include-exclude-principle)
\[
|\{h : h \text{ solution of (7)}\}| - |\{h : h \text{ solution of (7)}, \theta_1 = 0\}| - \cdots - |\{h : h \text{ solution of (7)}, \theta_\sigma = 0\}| + \cdots
\]
\[
= \sum_{j=0}^{\sigma - l} \binom{\sigma}{j} b^{\sigma - l - j} (-1)^j =: A(l).
\]
Hence
\[
\sum_{\substack{k_1 = b^{\sigma_1} + 1 \cdot \ldots \cdot \sum_{k_\sigma = b^{\sigma_\sigma} + 1 \cdot \sum_{\beta_{k_\sigma}, \beta_{u(k_\sigma)} = 1 = \beta
}}}}\beta \]
(ii) Assume that \(u \leq R_\sigma\). We rewrite system (7) in the form
\[
\begin{pmatrix}
c^{(1)}_{1,1} & \cdots & c^{(1)}_{a_1,1} & \cdots & c^{(1)}_{1,1} & \cdots & c^{(1)}_{a_\sigma,1} \\
c^{(1)}_{1,R_\sigma} & \cdots & c^{(1)}_{a_1,R_\sigma} & \cdots & c^{(1)}_{1,R_\sigma} & \cdots & c^{(1)}_{a_\sigma,R_\sigma} \\
c^{(1)}_{1,R_\sigma + \sigma} & \cdots & c^{(1)}_{a_1,R_\sigma + \sigma} & \cdots & c^{(1)}_{1,R_\sigma + \sigma} & \cdots & c^{(1)}_{a_\sigma,R_\sigma + \sigma}
\end{pmatrix}
\begin{pmatrix}
K_0^{(1)} \\
\vdots \\
K_{a_1 - 1}^{(1)} \\
K_0^{(2)} \\
\vdots \\
K_{a_\sigma - 1}^{(\sigma)}
\end{pmatrix}
= 
\begin{pmatrix}
(0, \ldots, 0, \beta, x_{u+1}, \ldots, x_{R_\sigma + \sigma})^\top - \sum_{j=1}^{\sigma} \theta_j c^{(j)}_{a,j+1}(R_\sigma + \sigma).
\end{pmatrix}
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Since the upper $R_\sigma \times R_\sigma$ sub-matrix of the above system is regular we find for arbitrary

$$x_{u+1}, \ldots, x_{R_\sigma} \quad \text{and} \quad \theta_1 \neq 0, \ldots, \theta_\sigma \neq 0$$

exactly one solution of the above subsystem. This solution can be made a solution of the whole system by an adequate choice of $x_{R_\sigma+1}, \ldots, x_{R_\sigma+\sigma}$. Therefore, we have

$$\sum_{k_1=b^a_1}^{b^a+1-1} \cdots \sum_{k_\sigma=b^\sigma_a}^{b^\sigma+1-1} 1 = (b-1)^\sigma b^{R_\sigma-u}.$$ 

Now we have

$$\Sigma(\{1, \ldots, \sigma\})$$

$$= \frac{12}{b^2(b^2-1)} \sum_{a_1, \ldots, a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{R_\sigma} \sum_{\beta=1}^{b^u-1} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{k_1=b^a_1}^{b^a+1-1} \cdots \sum_{k_\sigma=b^\sigma_a}^{b^\sigma+1-1} 1$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma \sum_{a_1, \ldots, a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} \sum_{u=1}^{R_\sigma} b^u \psi_b \left( \frac{N}{b^u} \right)$$

$$+ \frac{12}{b^2(b+1)} \sum_{a_1, \ldots, a_\sigma=0}^{\infty} b^{2l} \psi_b \left( \frac{N}{b^{R_\sigma+l}} \right) A(l)$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{l=1}^{\infty} b^{2l} A(l)$$

$$+ \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{l=1}^{\infty} 1$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{w=1}^{\infty} \left( w + \sigma - 1 \right) b^{-w}$$

$$+ \frac{12}{b^2(b+1)} \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{l=1}^{\infty} b^{2l} A(l) \left( u - l + \sigma - 1 \right) \left( u - l \right).$$
So we get
\[(NF_{b,N}(\omega))^2\]
\[= \frac{1}{(b+1)^s - 1} \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} \sum_{u \in \{w_1, \ldots, w_{|u|}\}} \Sigma(w_1, \ldots, w_{|u|})\]
\[= \frac{1}{(b+1)^s - 1} \sum_{v=1}^{s} \binom{s}{v} \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{w=u}^{\infty} \left( w + v - 1 \right) b^{-w}\]
\[= \sum_{u=1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) P_s(u).\]

This finishes the proof of 1. \(\square\)

Now let \(\omega\) be a digital \((t, s)\)-sequence over \(\mathbb{Z}_b\), \(t > 0\). Let us rewrite system (8) as

\[
\tilde{C} = \begin{pmatrix}
K_0^{(1)} \\
\vdots \\
K_{a_1 - 1}^{(1)} \\
K_0^{(\sigma)} \\
\vdots \\
K_{\alpha_{\sigma} - 1}^{(\sigma)}
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\beta \\
X_{u+1} \\
\vdots \\
X_{R_{\alpha} + \sigma + t}
\end{pmatrix} - \theta_1 \begin{pmatrix}
c_{a_1+1,1}^{(1)} \\
\vdots \\
c_{a_{\sigma}+1,1}^{(\sigma)}
\end{pmatrix} - \cdots - \theta_{\sigma} \begin{pmatrix}
c_{a_1+1,R_{\alpha}+\sigma+t}^{(1)} \\
\vdots \\
c_{a_{\sigma}+1,R_{\alpha}+\sigma+t}^{(\sigma)}
\end{pmatrix},
\]

(8)
where

$$\tilde{C} := \begin{pmatrix} c_{1,1}^{(1)} & \cdots & c_{a_1,1}^{(1)} & \cdots & c_{1,1}^{(\sigma)} & \cdots & c_{a_\sigma,1}^{(\sigma)} \\ c_{1,2}^{(1)} & \cdots & c_{a_1,2}^{(1)} & \cdots & c_{1,2}^{(\sigma)} & \cdots & c_{a_\sigma,2}^{(\sigma)} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{1,R_\sigma+\sigma+t}^{(1)} & \cdots & c_{a_1,R_\sigma+\sigma+t}^{(1)} & \cdots & c_{1,R_\sigma+\sigma+t}^{(\sigma)} & \cdots & c_{a_\sigma,R_\sigma+\sigma+t}^{(\sigma)} \end{pmatrix}.$$ 

Obviously, the matrix $\tilde{C}$ has rank $R_\sigma$. Let now $1 \leq u \leq R_\sigma + \sigma + t$ and $\beta \in \{1, \ldots, b-1\}$ be fixed. For a fixed choice of $x_{u+1}, \ldots, x_{R_\sigma+\sigma+t}$ and $\theta_1, \ldots, \theta_\sigma$, it is clear that we have at most one solution of system (8). Therefore we have

$$\sum_{k_1=b^\alpha+1}^{b^\alpha+1-1} \cdots \sum_{k_\alpha=b^{\alpha+1}}^{b^{\alpha+1}-1} 1 \leq (b-1)^\sigma b^{R_\sigma+\sigma+t-u}.$$ 

Now we have

$$\Sigma(\{1, \ldots, \sigma\})$$

$$\leq \frac{12}{b^2(b^2-1)} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma+t-1} b^{2u} \psi_b^\beta \left( \frac{N}{b^u} \right) (b-1)^\sigma b^{R_\sigma+\sigma+t-u}$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \sum_{a_1=0}^{\infty} \cdots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} \sum_{u=1}^{R_\sigma+\sigma+t} b^u \psi_b \left( \frac{N}{b^u} \right)$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \sum_{u=1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{a_1,\ldots,a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}}$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} \times \left( \sum_{u=1}^{\sigma+t} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{a_1,\ldots,a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} + \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{a_1,\ldots,a_\sigma=0}^{\infty} \frac{1}{b^{R_\sigma}} \right)$$

$$= \frac{12}{b^2(b+1)} (b-1)^\sigma b^{\sigma+t} (\Sigma_1 + \Sigma_2).$$
For $\Sigma_1$ we have
\[
\Sigma_1 = \sum_{u=1}^{\sigma+t} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{\substack{a_1, \ldots, a_{\sigma} = 0 \\ a_1 + \cdots + a_{\sigma} = 1}} 1 \\
\leq \frac{(b^2 - 1)(b+1)}{36} \left( \sum_{a=0}^{\infty} \frac{1}{b^a} \right) \sum_{u=1}^{\sigma+t} b^u \\
\leq \frac{(b+1)^2}{36} \left( \frac{b}{b-1} \right)^{\sigma+t+1}.
\]

For $\Sigma_2$ we have
\[
\Sigma_2 = \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{\substack{R_\sigma \geq u-t-\sigma}} 1 \\
= \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \sum_{w=u-t-\sigma}^{\infty} \frac{1}{b^w} \left( \frac{w + \sigma - 1}{\sigma - 1} \right).
\]

We now use [6 Lemma 6] to obtain
\[
\Sigma_2 \leq \sum_{u=\sigma+t+1}^{\infty} b^u \psi_b \left( \frac{N}{b^u} \right) \frac{1}{b^{u-t-\sigma}} \left( \frac{u-t-1}{\sigma - 1} \right) \left( \frac{b}{b-1} \right)^{\sigma} \\
= b^{t+\sigma} \left( \frac{b}{b-1} \right)^{\sigma} \sum_{u=\sigma+t+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) \left( \frac{u-t-1}{\sigma - 1} \right) \\
\leq b^{t+\sigma} \left( \frac{b}{b-1} \right)^{\sigma} \sum_{u=\sigma+t+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{\sigma-1}.
\]

This yields
\[
\Sigma(\{1, \ldots, \sigma\}) \leq \frac{12}{b^2(b+1)} (b-1)^{\sigma+\sigma+t} \\
\times \left( \frac{(b+1)^2}{36} \left( \frac{b}{b-1} \right)^{\sigma+t+1} + b^{t+\sigma} \left( \frac{b}{b-1} \right)^{\sigma} \sum_{u=\sigma+t+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{\sigma-1} \right) \\
= b^{3\sigma+2t} \left( \frac{b+1}{3b} + \frac{12}{b^2(b+1)} \sum_{u=\sigma+t+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{\sigma-1} \right).
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Hence

$$(NF_{b,N}(\omega))^2 = \frac{1}{(b+1)^s - 1} \sum_{\emptyset \neq u \subseteq \{1, \ldots, s\}} \Sigma_1\Sigma_2\left(\{w_1, \ldots, w_{|u|}\}\right)$$

$$\leq \frac{1}{(b+1)^s - 1} \sum_{v=1}^s \left(\begin{array}{c}s \\ v \end{array}\right) b^{2v+3v}$$

$$\times \left(\frac{b+1}{3b} + \frac{12}{b^2(b+1)} \sum_{u=t+v+1}^{\infty} \psi_b\left(\frac{N}{b^u}\right) (u-t-1)^{v-1}\right).$$

This finishes the proof of 2. \hfill \Box

We will need the following lemma due to Chaix and Faure [1, Lemme 6.3.1].

**Lemma 15.** Let $f$ be a real and bounded function; Set

$$d_n := \sup_{x \in \mathbb{R}} \left| \sum_{j=1}^{n} f(x/b^j) \right| \quad \text{and} \quad \alpha := \inf_{n \geq 1} \frac{d_n}{n};$$

then we have

$$\alpha = \lim_{n \to \infty} \frac{d_n}{n}.$$

**Proof of Corollary** [1] For any $N \leq b^m$ we have

$$(NF_{b,N}(\omega))^2 = \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) P_s(u) + \sum_{u=m+1}^{\infty} \frac{b^2(b^2-1)N^2}{12b^{2u}} P_s(u)$$

$$\leq \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) P_s(u) + c \sum_{u=m+1}^{\infty} \frac{u^{s-1}}{b^{2u}}$$

$$= \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) P_s(u) + \mathcal{O}(1).$$

From this it follows immediately that

$$F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log N)^{s/2}}{N}\right).$$

Let

$$P_s(u) := \sum_{i=0}^{s-1} \alpha_i u^i.$$
Then we have

\[
\frac{(NF_{b,N}(\omega))^2}{(\log N)^s} = \alpha_{s-1}\sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) \frac{u^{s-1}}{(\log N)^s} + \sum_{i=0}^{s-2} \alpha_i \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) \frac{u^i}{(\log N)^s} + \mathcal{O}\left(\frac{1}{(\log N)^s}\right).
\]

As \( N \) tends to infinity the term \( \mathcal{O}\left(\frac{1}{(\log N)^s}\right) \) goes to zero. For the second term we have

\[
\sum_{i=0}^{s-2} \alpha_i \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) \frac{u^i}{(\log N)^s} \leq \sum_{i=0}^{s-2} \alpha_i C_{\log N}^i (\log N)^{s-1} N \to \infty \to 0.
\]

So it is sufficient to consider only the first term. We have

\[
\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} = \sqrt{\alpha_{s-1} \limsup_{N \to \infty} \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) \frac{u^{s-1}}{(\log N)^s}}
\]

\[
\leq \sqrt{\frac{\alpha_{s-1}}{\log b} \limsup_{N \to \infty} \sum_{u=1}^{m} \psi_b\left(\frac{N}{b^u}\right) / \log N},
\]

with equality if \( s = 1 \). Let

\[
d_n := \sup_{x \in \mathbb{R}} \left| \sum_{u=1}^{n} \psi_b\left(\frac{x}{b^u}\right) \right|.
\]

\( \psi_b \) attains its maximum in a point \( \frac{k}{b^u} \). The function \( x \mapsto \sum_{u=1}^{m} \psi_b\left(\frac{x}{b^u}\right) \) is continuous and \( b^m \)-periodic, attains its maximum in a point \( N_m \) in \( [b^m, 2b^m) \), \( N_m \in \mathbb{N} \). So there exists a sequence of growing integers \( (N_m) \) such that

\[
\frac{d_m}{(m+1) \log b} \leq \sum_{u=1}^{m} \psi_b\left(\frac{N_m}{b^u}\right) / \log N_m \leq \frac{d_m}{m \log b}.
\]

So we get with Lemma 15

\[
\lim_{m \to \infty} \frac{\sum_{u=1}^{m} \psi_b\left(\frac{N_m}{b^u}\right)}{\log N_m} = \lim_{m \to \infty} \frac{d_m}{m \log b} = \inf_{m \geq 1} \frac{d_m}{m \log b} =: \frac{\gamma_b}{\log b}.
\]
Hence
\[
\limsup_{N \to \infty} \frac{NF_{b,N}(\omega)}{(\log N)^{s/2}} \leq \sqrt{\frac{\alpha_{s-1}}{(\log b)^s}}.
\]
\(\alpha_{s-1}\) is the leading coefficient from Remark 8. Since \(\psi_b\) is bounded by \(\frac{(b+1)(b^2-1)}{36}\) we have
\[
\gamma_b \leq \frac{(b+1)(b^2-1)}{36} < \infty.
\]
Now we have to compute \(\gamma_b\). We show that
\[
\gamma_b = \begin{cases} 
\frac{1}{9} & \text{if } b = 2, \\
\frac{b(b^2-1)}{48} & \text{if } b \text{ is odd.}
\end{cases}
\]
The case \(b = 2\) was already shown in [16, Corollary 2.3] since
\[
\limsup_{N \to \infty} \frac{NF_{2,N}(\omega)}{(\log N)^{1/2}} = \sqrt{\frac{1}{3 \log 2}}
\]
implies \(\gamma_2 = \frac{1}{9}\). It also follows from [1] Théorème 4.13 together with [4].
In the case \(b = 2p + 1\) for an integer \(p\), we can follow exactly the lines of the proof of [1] Théorème 4.13, since all properties of the function \(\chi_b^f\) used in this proof also hold for our function \(\psi_b\) (see Lemma [11] and Lemma [13]). One has only to use the values
\[
\psi_b \left( \frac{k}{b} \right) \quad \text{and} \quad \psi_b \left( \frac{1}{2} \right)
\]
instead of \(\chi_b^f \left( \frac{k}{b} \right) \quad \text{and} \quad \chi_b^f \left( \frac{1}{2} \right)\).

\[\square\]

Proof of Corollary 10 Suppose that \(b^{m-1} < N \leq b^m\) with \(m > t + s\), then
\[
\sum_{u=t+v+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{v-1} = \sum_{u=t+v+1}^{m} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{v-1} + \sum_{u=m+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u-t-1)^{v-1} =: \Sigma_3 + \Sigma_4.
\]
Now,
\[
\Sigma_3 \leq \frac{(b^2 - 1)(b + 1)}{36} \sum_{u=t+v+1}^{m} (u - t - 1)^{v-1}
\]
\[
\leq \frac{(b^2 - 1)(b + 1)}{36} (m - t - 1)^{v-1} \sum_{u=t+v+1}^{m} 1
\]
\[
\leq \frac{(b^2 - 1)(b + 1)}{36} (m - t - 1)^v
\]
and
\[
\Sigma_4 = \sum_{u=m+1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) (u - t - 1)^{v-1}
\]
\[
= \sum_{u=m+1}^{\infty} \frac{b^2(b^2 - 1)}{12} \left( \frac{N}{b^u} \right)^2 (u - t - 1)^{v-1}
\]
\[
= \sum_{u=1}^{\infty} \frac{b^2(b^2 - 1)}{12} \left( \frac{N}{b^u+m} \right)^2 (u + m - t - 1)^{v-1}
\]
\[
= \frac{b^2(b^2 - 1)}{12} \left( \frac{N}{b^u+m} \right)^2 \sum_{u=1}^{\infty} \frac{1}{b^{2u}} (u + m - t - 1)^{v-1}
\]
\[
\leq \frac{b^2(b^2 - 1)}{12} \sum_{u=1}^{\infty} \frac{1}{b^{2u}} \sum_{k=0}^{v-1} \left( \frac{v}{k} \right) (m - t - 1)^k u^{v-1-k}
\]
\[
\leq \frac{b^2(b^2 - 1)}{12} (m - t - 1)^{v-1} \sum_{u=1}^{\infty} \frac{1}{b^{2u}} \sum_{k=0}^{v-1} \left( \frac{v}{k} \right) 1^k u^{v-1-k}
\]
\[
= \frac{b^2(b^2 - 1)}{12} (m - t - 1)^{v-1} \sum_{u=1}^{\infty} \frac{(u + 1)^{v-1}}{b^{2u}}
\]
\[
= \frac{b^2(b^2 - 1)}{12} (m - t - 1)^{v-1} c_v.
\]
Hence
\[
(NF_{b,N}(\omega))^2
\]
\[
\leq \frac{1}{(b + 1)^s - 1} \sum_{v=1}^{s} \left( \binom{s}{v} \right) b^{2t+3v}
\]
\[
\times \left( \frac{b+1}{3b} + \frac{b^2 - 1}{3b^2} (m - t - 1)^v + (b - 1)(m - t - 1)^{v-1} c_v \right).
\]
The other results follow immediately. \(\square\)
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