THE $b$-ADIC DIAPHONY OF DIGITAL
$(T, s)$-SEQUENCES

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ABSTRACT. The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit cube. In this article we give an upper bound on the $b$-adic diaphony of digital $(T, s)$-sequences over $\mathbb{Z}_b$. And we derive a condition on the quality function $T$ such that the $b$-adic diaphony of the digital $(T, s)$-sequence over $\mathbb{Z}_b$ is of order $O((\log N)^s/2N-1)$. We also give a metrical result; for $\mu_s$-almost all generators of a digital $(T, s)$-sequence over $\mathbb{Z}_b$ the $b$-adic diaphony of the sequence is of order $O((\log \log N)^2(\log N)^{3s/2}N^{-1})$.

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1. Introduction

The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the $s$-dimensional unit cube. This notion was introduced by Hellekalek and Leeb \cite{6} for $b = 2$ and later generalized by Grozdanov and Stoilova \cite{5} for general integers $b \geq 2$. We recall now the definition of $b$-adic Walsh functions, which will be needed for the definition of the $b$-adic diaphony.

Let $b \geq 2$ be an integer. For a nonnegative integer $k$ with base $b$ representation $k = \kappa_{a-1}b^{a-1} + \cdots + \kappa_1b + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$ and $\kappa_{a-1} \neq 0$, we define the Walsh function $b_{\text{wal}}(x) : [0, 1) \to \mathbb{C}$ by

$$b_{\text{wal}}(x) := e^{2\pi i(x_1\kappa_0 + \cdots + x_{a-1}\kappa_{a-1})/b},$$

for $x \in [0, 1)$ with base $b$ representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the $x_i$ must be different from $b-1$).
For higher dimensions $s \geq 1$, $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in [0,1)^s$ we write

$$b_{\text{wal}}(x) := \prod_{j=1}^s b_{\text{wal}}(x_j).$$

Now we are ready to define the $b$-adic diaphony (see [5] or [6]).

**Definition 1.** Let $b \geq 2$ be an integer. The $b$-adic diaphony of the first $N$ elements of a sequence $\omega = (x_n)_{n \geq 0}$ in $[0,1)^s$ is defined by

$$F_{b,N}(\omega) := \left( \frac{1}{(1+b)^s - 1} \sum_{\substack{k \in \mathbb{N}_0^s \\mid k \neq 0}} r_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} b_{\text{wal}}(x_n) \right| \right)^{1/2},$$

where for $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, $r_b(k) := \prod_{j=1}^s r_b(k_j)$ and for $k \in \mathbb{N}_0$,

$$r_b(k) := \begin{cases} 
1 & \text{if } k = 0 \\
b^{-2a} & \text{if } b^a \leq k < b^{a+1} \text{ where } a \in \mathbb{N}_0.
\end{cases}$$

Throughout this article we will write $a(k) = a$, if $a$ is the unique determined integer such that $b^a \leq k < b^{a+1}$. If $b = 2$ we also speak of dyadic diaphony.

The $b$-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence $\omega$ in the $s$-dimensional unit cube is uniformly distributed modulo one if and only if $\lim_{N \to \infty} F_{b,N}(\omega) = 0$. This was shown in [6] for the case $b = 2$ and in [5] for the general case. Further it is shown in [1] that the $b$-adic diaphony is – up to a factor depending on $b$ and $s$ – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space $H_{\text{wal},s,\gamma}$, which has been introduced in [2].

In the following let $b$ be a prime, i.e. we can always take $\mathbb{Z}_b$ for the finite field of prime order $b$. We consider the $b$-adic diaphony of digital $(T,s)$-sequences over $\mathbb{Z}_b$. Here $s$ is the dimension of the sequence and $T : \mathbb{N}_0 \to \mathbb{N}_0$ is the quality function of the sequence; lower quality functions imply stronger equidistribution properties. A special class among these functions are the digital $(t,s)$-sequences over $\mathbb{Z}_b$, where the quality function $T$ is a constant $t$. Digital $(t,s)$-sequences were introduced by Niederreiter [8, 9]. The concept of $(T,s)$-sequences was introduced by Larcher and Niederreiter in [7], as a quality function $T$ is a more sensitive measure than a quality parameter $t$. For more information on $(T,s)$-sequences see [3, Chapter 4].

In [4] the author showed a formula for the $b$-adic diaphony of digital $(0,s)$-sequences over $\mathbb{Z}_b$, $s = 1,\ldots,b$, and an upper bound for the $b$-adic diaphony of
THE $b$-ADIC DIAPHONY OF DIGITAL $(T,s)$-SEQUENCES

digital $(t,s)$-sequences over $\mathbb{Z}_b$ for primes $b$. In both cases we obtained for the asymptotic order

$$F_{b,N}(\omega) = O\left(\frac{(\log N)^{s/2}}{N}\right) \quad \text{(as } N \to \infty)$$

(1)

In this article we would like now to find in analogy to [4] an upper bound on the $b$-adic diaphony of digital $(T,s)$-sequences over $\mathbb{Z}_b$ and give a condition on the quality function $T$, so that we obtain the same order as in (1). With a weaker condition on $T$ we still obtain the asymptotic order

$$(NF_{b,N}(\omega))^2 = O\left(\sum_{u=1}^{[\log_b N]} u^{s-1} b^2 T(u)\right) \quad \text{(as } N \to \infty).$$

Now we give a definition of digital $(T,s)$-sequences over $\mathbb{Z}_b$. The quality function $T$ is closely related to a quantity $\rho_m$, which in some sense “measures” the “linear independence” of $s$ infinite matrices $C_1, \ldots, C_s$ (see [3, Chapter 4.4]). Let $C_1, \ldots, C_s$ be $N \times N$ matrices over the finite field $\mathbb{Z}_b$. For any integers $1 \leq i \leq s$ and $m \geq 1$ by $C_i(m)$ we denote the left upper $m \times m$ sub-matrix of $C_i$. Then

$$\rho_m = \rho_m(C_1, \ldots, C_s) := \rho(C_1^{(m)}, \ldots, C_s^{(m)}),$$

where $\rho$ is the independence parameter defined for $s$-tuples of $m \times m$ matrices over $\mathbb{Z}_b$, i.e. $\rho$ is the largest integer such that for any choice $d_1, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = \rho$, the following holds:

- the first $d_1$ row vectors of $C_1^{(m)}$ together with
- the first $d_2$ row vectors of $C_2^{(m)}$ together with
  
  $\vdots$

- the first $d_s$ row vectors of $C_s^{(m)}$ are linearly independent over the finite field $\mathbb{Z}_b$.

**Definition 2.** For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base $b$ representation of $n$. For $j \in \{1, \ldots, s\}$ multiply the vector $\mathbf{n} = (n_0, n_1, \ldots)^\top$ by the matrix $C_j$,

$$C_j \cdot \mathbf{n} =: (x_n^j(1), x_n^j(2), \ldots)^\top \in \mathbb{Z}_b^\infty,$$

and set

$$x_n^{(j)} := \frac{x_n^j(1)}{b} + \frac{x_n^j(2)}{b^2} + \cdots.$$

Finally set $\mathbf{x}_n := (x_n^{(1)}, \ldots, x_n^{(s)})$. 

3
The digital sequence $\omega$ constructed this way by the $N \times N$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_b$ is a strict $(T, s)$-sequence in base $b$ with $T(m) = m - \rho_m$ for all $m \in \mathbb{N}$. The matrices $C_1, \ldots, C_s$ are called the generator matrices of the sequence.

**Remark 1.** (1) Any strict digital $(T, s)$-sequence over $\mathbb{Z}_b$ is a digital $(U, s)$-sequence over $\mathbb{Z}_b$ for all $U$ with $U(m) \geq T(m)$ for all $m$.

(2) The concept of $(t, s)$-sequences in base $b$ is contained in the concept of $(T, s)$-sequences in base $b$. We just have to take for $T$ the constant function $T(m) = t$ for all $m$ (resp. $T(m) = m$ for $m \leq t$).

For more information on digital $(T, s)$-sequences we refer to [3].

**Definition 3.** Let $\omega$ be a uniformly distributed strict digital $(T, s)$-sequence in base $b$. For $r \in \mathbb{N}_0$ we set
\[ \eta(r) := \min\{m : m - T(m) \geq r\}. \]
This minimum exists for all $r$, because $\lim_{m \to \infty} m - T(m) = \infty$ if $\omega$ is uniformly distributed modulo 1 (see [3, Theorem 4.32]).

We will need the following properties of the function $\eta$:

(1) $\eta$ is non-decreasing.

(2) The condition $\eta(r) > u$ is equivalent to $u - T(u) < r$.

These properties follow easily from the fact that $S(m) := m - T(m)$ is non-decreasing (see [3, p.133]) and from the definition of $\eta$.

Finally we need the definition of the function $\psi_b$. Let $\beta$ be an integer in $\{1, \ldots, b - 1\}$. For $x \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$, $j \in \{0, \ldots, b - 1\}$ we set
\[ \psi_{b}^\beta(x) := \frac{b^2(b^2 - 1)}{12} \left| 1 - \frac{z_{\beta}^j}{b z_{\beta} - 1} + \frac{z_{\beta}^j}{b} \left(x - \frac{j}{b}\right) \right|^2, \]
where $z_{\beta} = e^{\frac{2\pi i}{b} \beta} = \text{wal}_{1}\left(\frac{\beta}{b}\right)$; then the function is extended to the reals by periodicity. The function $\psi_b$ is now defined as the mean of the functions $\psi_{b}^\beta$:
\[ \psi_b(x) := \frac{1}{b - 1} \sum_{\beta=1}^{b-1} \psi_{b}^\beta(x). \]

We will need two facts about $\psi_b$:

(1) The function $\psi_b$ is bounded (see [4, Lemma 12]),

4
THE $b$-ADIC DIAPHONY OF DIGITAL $(T, s)$-SEQUENCES

(2) $\psi_b(x) = \frac{b^2(b^2-1)}{12} x^2$ on the interval $[0, \frac{1}{b})$ (see [4, Lemma 11(1)]).

For further properties of the function $\psi_b$ we refer to [4, Lemma 11, Lemma 13].

2. Results

We show now an upper bound on the $b$-adic diaphony of digital $(T, s)$-sequences. From this we derive (under certain conditions on the quality function $T$) the asymptotic order of the $b$-adic diaphony of these sequences. We also give a metrical result. The proofs of the results below are given in Section 3.

**Theorem 1.** Let $\omega$ be a digital $(T, s)$-sequence over $\mathbb{Z}_b$. For any $N \geq 1$ we have

\[
(NF_{b,N}(\omega))^2 \leq \frac{1}{(b+1)^s - 1} \frac{12}{b^3(b+1)} \sum_{w=1}^{s} \binom{s}{w} \left( \frac{b^4}{b^2 - 1} \right)^w \sum_{u=1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) b^u \sum_{v=u-1}^{\infty} \frac{v^{w-1} b^{2T(v)}}{b^v},
\]

where $c$ is a constant that does not depend on $N$.

From the above theorem we obtain now the asymptotic behaviour of certain digital $(T, s)$-sequences over $\mathbb{Z}_b$.

**Corollary 1.** Let $\omega$ be a digital $(T, s)$-sequence over $\mathbb{Z}_b$ satisfying the property that

\[
\sum_{v=u-1}^{\infty} \frac{v^{s-1} b^{2T(v)}}{b^v} \leq c_1 \frac{u^{s-1} b^{2T(u)}}{b^u} \quad \text{for all } u \in \mathbb{N},
\]

where $c_1$ is a constant that does not depend on $u$. Then for the $b$-adic diaphony of the first $N \geq 2$ elements of $\omega$ we have

\[
(NF_{b,N}(\omega))^2 = \mathcal{O} \left( \sum_{u=1}^{[\log_b N]} u^{s-1} b^{2T(u)} \right) \quad (\text{as } N \to \infty).
\]

**Remark 2.** (1) For $u \in \mathbb{N}, s \in \mathbb{N}_0$ we have

\[
\sum_{v=u-1}^{\infty} \frac{v^s}{b^v} \leq \left( 2b \sum_{v=0}^{\infty} \frac{v^s}{b^v} \right) \frac{u^s}{b^u} \leq c_1 \frac{u^s}{b^u}.
\]
For $T(v) = t$ the above condition is satisfied and we get immediately the asymptotic order of the $b$-adic diaphony of digital $(t, s)$-sequences over $\mathbb{Z}_b$

$$F_{b,N}(\omega) = O\left(\frac{(\log N)^{s/2}}{N}\right) \quad (\text{as } N \to \infty),$$

(see also [4, Corollary 10]).

(3) For $T(m) \leq \max(C, \log_b \log m)$, $C \geq 0$, the above condition is satisfied and we get

$$F_{b,N}(\omega) = O\left(\frac{(\log \log N)(\log N)^{s/2}}{N}\right) \quad (\text{as } N \to \infty).$$

In the last example we already came close to the desired asymptotic order $O((\log N)^{s/2}N^{-1})$. In the next corollary we give an additional condition on $T$, which guarantees such an asymptotic behaviour.

**Corollary 2.** Let $\omega$ be a digital $(T, s)$-sequence over $\mathbb{Z}_b$ satisfying

1. $\sum_{v=u-1}^{\infty} \frac{u_{b^s-1}}{b^s} b^{2T(v)} \leq c_1 \frac{u_{b^s-1}}{b^s} b^{2T(u)}$ for all $u \in \mathbb{N}$,
2. $\frac{1}{m} \sum_{u=1}^{m} b^{2T(u)} \leq c_2$ for all $m \in \mathbb{N}$,

where the constants $c_1, c_2$ do not depend on $u$ and $m$, respectively. Then we have

$$F_{b,N}(\omega) = O\left(\frac{(\log N)^{s/2}}{N}\right) \quad (\text{as } N \to \infty).$$

Now we are interested in the order of the $b$-adic diaphony of digital $(T, s)$-sequences, when the quality function $T$ does not necessarily fulfill the conditions from Corollary 2 or Corollary 1, i.e. what order we can get for almost all digital $(T, s)$-sequences. In the following we explain what we mean by “almost all”.

Let $\mathcal{M}_s$ denote the set of all $s$-tuples of $\mathbb{N} \times \mathbb{N}$ matrices over $\mathbb{Z}_b$. We define the probability measure $\mu_s$ on $\mathcal{M}_s$ as the product measure induced by a certain probability measure $\mu$ on the set $\mathcal{M}$ of all infinite matrices over $\mathbb{Z}_b$. We can view $\mathcal{M}$ as the product of denumerable many copies of the sequence space $\mathbb{Z}_b^\mathbb{N}$ over $\mathbb{Z}_b$, and so we define $\mu$ as the product measure induced by a certain probability measure $\tilde{\mu}$ on $\mathbb{Z}_b^\mathbb{N}$. For $\tilde{\mu}$ we just take the measure on $\mathbb{Z}_b^\mathbb{N}$ induced by the equiprobability measure on $\mathbb{Z}_b$.

We use now the result from [3, Example 5.50.], that $\mu_s$-almost all $s$-tuples $(C_1, \ldots, C_s) \in \mathcal{M}_s$ generate a digital $(T, s)$-sequence over $\mathbb{Z}_b$ such that for some constant $L$ we have

$$T(m) \leq s \log_b m + 2 \log_b \log m + L$$

(2)
for all integers \( m \geq 2 \), to obtain the following metrical result as a consequence of Corollary 1.

**Corollary 3.** \( \mu_s \)-almost all \( s \)-tuples \((C_1, \ldots, C_s) \in \mathcal{M}_s \) generate a digital \((T, s)\)-sequence over \( \mathbb{Z}_b \) such that

\[
F_{b,N}(\omega) = \mathcal{O}\left(\frac{(\log \log N)^2(\log N)^{3s/2}}{N}\right) \quad \text{(as } N \to \infty)\text{.}
\]

### 3. Proofs

In this section we provide now the proofs of the previous results from Section 2.

**Proof of Theorem 1** It is enough to show Theorem 1 for strict digital \((T, s)\)-sequences over \( \mathbb{Z}_b \). If \( \omega \) is not uniformly distributed modulo one the upper bound in Theorem 1 is infinite and therefore trivially fulfilled. So let in the following \( \omega \) be a uniformly distributed, strict digital \((T, s)\)-sequence over \( \mathbb{Z}_b \), i.e. the function \( \eta \) is always well defined. The first steps of this proof are the same as in [4, Proof of Theorem 6]. So we just recall these steps without a detailed elaboration. For a point \( x_n \) of \( \omega \) and for \( \emptyset \neq u \subset \{1, \ldots, s\} \), we define \( x_n^{(u)} \) as the projection of \( x_n \) onto the coordinates in \( u \). We have

\[
(NF_{b,N}(\omega))^2 = \frac{1}{(b+1)^s - 1} \sum_{\emptyset \neq u \subset \{1, \ldots, s\}} \Sigma(u),
\]

where

\[
\Sigma(u) := \sum_{k_{w_1}=1}^{\infty} \cdots \sum_{k_{w_{|u|}}=1}^{\infty} \left( \prod_{j \in u} \frac{1}{b^{2a(k_j)}} \right) \left| \sum_{n=0}^{N-1} b \text{wal}_{(k_{w_1}, \ldots, k_{w_{|u|}})}(x_n^{(u)}) \right|^2.
\]

For the sake of simplicity we assume in the following \( u = \{1, \ldots, \sigma\} \), \( 1 \leq \sigma \leq s \), and set \( k_\sigma := (k_1, \ldots, k_\sigma) \), where \( k_j, 1 \leq j \leq \sigma \), has \( b \)-adic expansion \( k_j = \kappa_0^{(j)} + \kappa_1^{(j)} b + \cdots + \kappa_\sigma^{(j)} b^\sigma, \kappa_\sigma^{(j)} \neq 0 \). The other cases are dealt with a similar fashion. Let \( C_j = (c_v^{(j)})_{v, w \geq 1} \) and let \( c_i^{(j)} \) be the \( i \)-th row vector of the generator matrix \( C_j \). Define

\[
u(k_\sigma) := \min \left\{ l \geq 1 : \sum_{j=1}^{\sigma} (\kappa_0^{(j)} c_{1,j}^{(j)} + \cdots + \kappa_\sigma^{(j)} c_{l+1,j}^{(j)}) \neq 0 \right\}
\]
and

$$\beta_{k_\sigma} = (\beta_{k_\sigma,0}, \beta_{k_\sigma,1}, \ldots)^\top := \sum_{j=1}^{\sigma} (\kappa_0^{(j)} c_1^{(j)} + \cdots + \kappa_{a_j}^{(j)} c_{a_j+1}^{(j)}).$$

Since $C_1, \ldots, C_\sigma$ generate a digital $(T, s)$-sequence over $\mathbb{Z}_b$ one can verify with the same arguments as in [4, Proof of Theorem 6] that $u(k_\sigma) \leq \eta \left( \sum_{j=1}^{\sigma} a_j + \sigma \right) =: \eta(R_\sigma + \sigma)$, since the $\eta(R_\sigma + \sigma) \times (R_\sigma + \sigma)$ matrix

$$C(a_1, \ldots, a_\sigma)$$

has rank $R_\sigma + \sigma$. We have

$$\Sigma(\{1, \ldots, \sigma\})$$

$$= \frac{12}{b^2 (b^2 - 1)} \sum_{a_1=0}^{\infty} \ldots \sum_{a_\sigma=0}^{\infty} \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{\eta(R_\sigma + \sigma) - b-1} b^{2u} \psi_b \left( \frac{N}{b^u} \right) \sum_{k_1=b^{a_1}}^{b^{a_1+1}-1} \cdots \sum_{k_\sigma=b^{a_\sigma}}^{b^{a_\sigma+1}-1} 1.$$

We need to evaluate the sum

$$\sum_{\substack{k_1=b^{a_1} \\ldots \kappa_\sigma=b^{a_\sigma} \\beta_{k_\sigma} = u \\beta_{k_\sigma} \neq u \\beta_{k_\sigma} = \beta}}^{b^{a_1+1}-1} \sum_{\substack{k_1=b^{a_1} \\ldots \kappa_\sigma=b^{a_\sigma} \\beta_{k_\sigma} = u \\beta_{k_\sigma} \neq u \\beta_{k_\sigma} = \beta}}^{b^{a_\sigma+1}-1} 1$$

for $1 \leq u \leq \eta(R_\sigma + \sigma)$ and $\beta \in \{1, \ldots, b - 1\}$. This is the number of digits $\kappa_0^{(1)}, \ldots, \kappa_{a_1-1}^{(1)}, \theta_1, \ldots, \kappa_0^{(\sigma)}, \ldots, \kappa_{a_\sigma-1}^{(\sigma)}, \theta_\sigma \in \{0, \ldots, b - 1\}$, $\theta_1 \neq 0, \ldots, \theta_\sigma \neq 0,$
such that

\[
C(a_1, \ldots, a_\sigma) = \left( \begin{array}{c}
\kappa_0^{(1)} \\
\vdots \\
\kappa_{a_1-1}^{(1)} \\
\theta_1 \\
\vdots \\
\kappa_{\sigma}^{(\sigma)} \\
\vdots \\
\kappa_{a_\sigma-1}^{(\sigma)} \\
\theta_{\sigma} \\
\end{array} \right) = \left( \begin{array}{c}
0 \\
\vdots \\
0 \\
\beta \\
\vdots \\
\beta \\
\end{array} \right)
\]

for arbitrary \(x_{u+1}, \ldots, x_{\eta(R_\sigma + \sigma)} \in \mathbb{Z}_b\). Let now \(1 \leq u \leq \eta(R_\sigma + \sigma)\) and \(\beta \in \{1, \ldots, b-1\}\) be fixed. For a fixed choice of \(x_{u+1}, \ldots, x_{\eta(R_\sigma + \sigma)}\) the system has at most one solution. There are \(b^u \beta^{u(R_\sigma + \sigma) - u}\) possible choices for the \(x_{u+1}, \ldots, x_{\eta(R_\sigma + \sigma)}\). So we have

\[
\sum_{k_1=b^u+1} b^{u+1} \cdots \sum_{k_\sigma=b^{u_\sigma}-1} b^{u_\sigma+1} 1 \leq b^{\eta(R_\sigma + \sigma) - u}.
\]

Now we have

\[
\Sigma(\{1, \ldots, \sigma\})
\]

\[
\leq \frac{12}{b^2(b^2-1)} \sum_{a_1, \ldots, a_{\sigma}=0}^\infty \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{\eta(R_\sigma + \sigma)} b^{u} \psi_b \left( \frac{N}{b^u} \right) b^{\eta(R_\sigma + \sigma) - u}
\]

\[
= \frac{12}{b^2(b+1)} \sum_{a_1, \ldots, a_{\sigma}=0}^\infty \frac{1}{b^{2R_\sigma}} \sum_{u=1}^{\eta(R_\sigma + \sigma)} \psi_b \left( \frac{N}{b^u} \right) b^{u} b^{\eta(R_\sigma + \sigma)}
\]

\[
= \frac{12}{b^2(b+1)} \sum_{a_1, \ldots, a_{\sigma}=0}^\infty \psi_b \left( \frac{N}{b^u} \right) \sum_{u=1}^{\eta(R_\sigma + \sigma)} b^{u} b^{\eta(R_\sigma + \sigma)}
\]

\[
= \frac{12}{b^2(b+1)} \sum_{a_1, \ldots, a_{\sigma}=0}^\infty \psi_b \left( \frac{N}{b^u} \right) \sum_{(u-1)-T(u-1)<R_\sigma + \sigma} b^{u} b^{\eta(R_\sigma + \sigma)}
\]
Proof of Corollary 1. For any $b^{m} < N \leq b^{m+1}$ we obtain out of Theorem 1 and the special form of $\psi_b$ on $[0, \frac{1}{b}]$ that

$$(NF_{b,N}(\omega))^2 \leq c \sum_{u=1}^{\infty} \psi_b \left( \frac{N}{b^u} \right) b^u \sum_{v=u-1}^{\infty} \frac{v^{s-1}}{b^v} b^2 T(v)$$

$$(NF_{b,N}(\omega))^2 \leq cc_1 \sum_{u=1}^{m} \psi_b \left( \frac{N}{b^u} \right) u^{s-1} b^2 T(u) + cc_1 \sum_{u=m+1}^{\infty} \frac{b^2(b^2 - 1)}{b^{2u}} N^2 u^{s-1} b^2 T(u)$$

$$(NF_{b,N}(\omega))^2 \leq cc_1 \sum_{u=1}^{m} \psi_b \left( \frac{N}{b^u} \right) u^{s-1} b^2 T(u) + cc_1 \frac{b^2(b^2 - 1)}{12} N^{m+1} b^m \sum_{u=m+1}^{\infty} \frac{u^{s-1}}{b^u} b^2 T(u)$$

$$(NF_{b,N}(\omega))^2 \leq c_1 \sum_{u=1}^{m} u^{s-1} b^2 T(u) + c_2 m^{s-1} b^2 T(m)$$
THE \( b \)-ADIC DIAPHONY OF DIGITAL \((T, s)\)-SEQUENCES

\[
\mathcal{O} \left( \sum_{u=1}^{m} u^{s-1} b^{2T(u)} \right),
\]

where all appearing constants may depend only on \( b \) and \( s \). \( \square \)

**Proof of Corollary 2.** From Corollary 1 and the additional condition that
\[
\frac{1}{m} \sum_{u=1}^{m} b^{2T(u)} \leq c_2 \text{ for all } m \in \mathbb{N},
\]
we get for any \( b^{m-1} < N \leq b^m \)

\[
(N F_{b,N}(\omega))^2 \leq \tilde{c} \sum_{u=1}^{m} u^{s-1} b^{2T(u)}
\]

\[
\leq \tilde{c} m^s \frac{1}{m} \sum_{u=1}^{m} b^{2T(u)}
\]

\[
\leq \tilde{c} c_2 m^s,
\]

where all appearing constants may depend only on \( b \) and \( s \). From this it follows immediately that

\[
F_{b,N}(\omega) = \mathcal{O} \left( \left( \frac{\log N}{N} \right)^{s/2} \right) \quad (\text{as } N \to \infty).
\]

\( \square \)

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