SOME METRICAL RESULTS ON THE APPROXIMATION BY CONTINUED FRACTIONS

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ABSTRACT. Let $x$ be a real irrational number and $p_n/q_n$, $n = 1, 2, \ldots$ its sequence of continued fraction convergents. Define $d_n = |q_{n+1}x - p_n|$. For almost all $x$ the distribution functions of the sequences $|d_n - d_{n+1}|$ and $d_n + d_{n+1}$ are determined.

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1. Introduction

Let $x$ be a real irrational number with continued fraction expansion

$$x = [a_0; a_1, a_2, \ldots] \quad \text{and} \quad p_n/q_n, \quad n = 1, 2, \ldots$$

its sequence of convergents.

The sequence $\theta_n$, $n = 1, 2, \ldots$ of approximation coefficients of $x$ is defined by

$$\theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \quad n = 1, 2, \ldots \quad (1)$$

One of the most important aspects of the approximation by continued fractions is the fact that this is always a sequence in the unit interval.

At the basis of many results on the distribution of these coefficients $\theta_n$ lies the following fundamental metrical result.

THEOREM 1. Denote by $\Delta$ the unit triangle in the $(\alpha, \beta)$-plane, that is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

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For every $x$ the sequence $(\theta_n, \theta_{n+1})$ is a sequence in $\Delta$ and for almost all $x$ it is distributed over $\Delta$ according to the density function $\mu$, where

$$
\mu(\alpha, \beta) = \frac{1}{\log 2} \frac{1}{\sqrt{1 - 4\alpha\beta}}.
$$

For a proof see [2, section 5.3] or [3].

From Theorem 1 a proof of the Doeblin-Lenstra conjecture on the distribution of the sequences $\theta_n$, $n = 1, 2, \ldots$, see [1], follows easily. Also the distribution functions, for almost all $x$, of the sequences

$$
\theta_n + \theta_{n+1}, \quad |\theta_n - \theta_{n+1}| \quad \text{and} \quad \theta_n \theta_{n+1}, \quad n = 1, 2, \ldots
$$

can be derived from it, see [3].

Less is known about another type of approximation coefficients, the sequences $d_n$, $n = 1, 2, \ldots$, defined by

$$
d_n = q_n q_{n+1} \left| x - \frac{p_n}{q_n} \right|, \quad n = 1, 2, \ldots \quad (2)
$$

For every irrational $x$ this is a sequence in the interval $[\frac{1}{2}, 1]$ with, for almost all $x$, a distribution function $F$ where

$$
F(z) = \frac{1}{\log 2} \left( z \log z + (1 - z) \log(1 - z) + \log 2 \right), \quad (3)
$$

see [1, Theorem 4].

The sequences from (1) and (2) are related by the fact that the two-dimensional sequence

$$(\theta_n, d_n), \quad n = 1, 2, \ldots,$$

is for all irrational $x$ a sequence in the interior of the triangle in the $(\alpha, \beta)$-plane with vertices $(0, 1)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(1, 1)$ and that for almost all $x$ this sequence is distributed over this triangle according to the density function

$$
\frac{1}{\log 2} \frac{1}{\alpha}, \quad (4)
$$

see [3, Theorem 7].

The distribution (4) also yields an easy proof of the Doeblin-Lenstra conjecture. Other consequences of (4) are the distribution (3) of the $d_n$’s and the uniform distribution in the unit interval of the sequence $d_n - \theta_n$, $n = 1, 2, \ldots$ see [3, Theorem 9]. Hence the mean of the $d_n$’s, $\mathcal{M}(d_n)$, differs $\frac{1}{2}$ from $\mathcal{M}(\theta_n)$. As $\mathcal{M}(\theta_n) = \frac{1}{4 \log 2}$, see [1, Corollary 2], we thus have

$$
\mathcal{M}(d_n) = \frac{1}{2} + \frac{1}{4 \log 2} = 0.86067 \ldots, \quad \text{a.e.}
$$

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By a simple transformation of the Doeblin-Lenstra conjecture one obtains the
distribution function of the sequences \( \log \theta_n \), \( n = 1, 2, \ldots \), over the interval
\( (-\infty, 0) \) and with this one easily shows that
\[
\mathcal{M}(\log \theta_n) = -1 - \frac{1}{2} \log 2 = -1.34657 \ldots, \ a.e.
\] (5)
The step to
\[
\mathcal{M}(\log d_n) = -1 - \frac{1}{2} \log 2 + \frac{\pi^2}{12 \log 2} = -0.16000 \ldots, \ a.e.,
\] then follows from Lévy’s celebrated result
\[
\lim_{n \to \infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2}, \ a.e.,
\] because
\[
\log d_n - \log \theta_n = \log q_{n+1} - \log q_n.
\]
To obtain more information about the distribution properties of the sequences
from (2) one would like a result similar to Theorem 1. This was given in [4] and
reads as follows.

**Theorem 2.** Let \( a \) be a positive integer and denote by \( R_a \) the quadrangle in the
\((\alpha, \beta)\)-plane with vertices
\[
(1, 1), \ (a/(a+1), 1), \ ((a+1)/(a+2), (a+1)/(a+2)) \quad \text{and} \quad (1, a/(a+1)).
\]
Consider the set \( \Omega \) as the union of the \( R_a \), \( a = 1, 2, \ldots \), piled one upon another,
where the edge from
\[
((a+1)/(a+2), (a+1)/(a+2)) \quad \text{to} \quad (1, a/(a+1)) \quad \text{of} \ R_a
\]
is identified, by vertical projection, with the edge from
\[
((a+1)/(a+2), 1) \quad \text{to} \quad (1, 1) \quad \text{of} \ R_{a+1}.
\]
Then, for every \( x \), the sequence \( (d_n, d_{n+1}) \) \( n = 1, 2, \ldots \) lies in \( \Omega \); more precisely
\[
(d_n, d_{n+1}) \in R_a \quad \text{if and only if} \quad a_{n+2} = a.
\]
For almost all \( x \) the sequence is distributed over \( \Omega \) according to the density
function \( \nu \), where
\[
\nu(\alpha, \beta) = \frac{1}{\log 2} \frac{1}{\alpha + \beta - 1}.
\]
A consequence of the first part of this theorem is that when a \( d_n \) is small, i.e.,
close to the left end point of the interval \([\frac{1}{2}, 1]\), its successor \( d_{n+1} \) is close to the
other end point of the interval. In the case of the coefficients \( \theta_n \) the situation is
just the opposite. When a $\theta_n$ is close to 1, its successor, and also its predecessor, are close to 0. Therefore it is of interest to study the distribution of the sequences

$$|\theta_n - \theta_{n+1}| \text{ and } |d_n - d_{n+1}|, \ n = 1, 2, \ldots$$

(6)

For the first sequence of (6) this was done in [3]. One has for instance for almost all $x$:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\theta_n - \theta_{n+1}| = \frac{4 - \pi}{4 \log 2} = 0.30960 \ldots$$

(7)

The only application of Theorem 2 in [4] was an alternative proof of (3). The object of this paper is to use Theorem 2 to obtain the distribution function for the second sequence of (6) and a result similar to (7) for the $d_n$. Further we determine the distribution of the sequences $d_n + d_{n+1}, \ n = 1, 2, \ldots$, for almost all $x$.

2. The distribution of the sequence $|d_n - d_{n+1}|$

Theorem 3. Put

$$m = m(z) = \left\lfloor \frac{1}{z} \right\rfloor, \ z > 0,$$

and define the function $F$ on the interval $[0, \frac{1}{2}]$ by

$$F(z) = \frac{1}{\log 2} \left( z \log \frac{m(m+1)}{2} - (m-1)((1+z)\log(1+z) + (1-z)\log(1-z)) ight. + \log \left( 1 + \frac{1}{m} \right) , \quad 0 < z \leq \frac{1}{2},$$

$$F(0) = 0.$$

This function $F$ is monotonically increasing from 0 in $z = 0$ to 1 in $z = \frac{1}{2}$ and has a continuous derivative.

It is for almost all $x$ the distribution function of the sequence

$$|d_n - d_{n+1}|, \ n = 1, 2, \ldots$$

The form of $F$ looks perhaps unpleasant but when one plots the graph, one gets a nice, smooth curve, see Figure 1.
Figure 1. Distribution function of $|d_n - d_{n+1}|$.

**Examples.**

\[
F\left(\frac{1}{4}\right) = \frac{41}{4} - \frac{1}{\log 2} \left(\frac{9}{4} \log 3 + \frac{5}{2} \log 5\right) = 0.87901 \ldots ,
\]

\[
F'\left(\frac{1}{4}\right) = 1 + \frac{1}{\log 2} \log \frac{27}{25} = 1.11103 \ldots ,
\]

\[
F\left(\frac{1}{8}\right) = \frac{157}{4} - \frac{1}{\log 2} \left(\frac{27}{2} \log 3 + \frac{49}{8} \log 7\right) = 0.65795 \ldots
\]

**Proof.** Let $a$ be a positive integer. Denote by $R_a(z)$, with $0 < z < 1/(a + 1)$, that part of $R_a$ for which $|\alpha - \beta| < z$. Then, in view of Theorem [2] one has, for almost all $x$, that

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n; n \leq N, |d_n - d_{n+1}| < z, a_{n+2} = a \} = \nu(R_a(z)).
\]
Denote this expression by $F_a(z)$. After a tedious, but otherwise completely elementary calculation, one finds that for $0 < z < 1/(a + 1)$

$$F_a(z) = \frac{1}{\log 2} \left( z \log \frac{a + 2}{a} - (1 + z) \log(1 + z) - (1 - z) \log(1 - z) \right).$$

Define $F_a(z)$ on the interval $[1/(a + 1), 1/2]$ by

$$F_a(z) = F_a \left( \frac{1}{a + 1} \right) = \frac{1}{\log 2} \log \frac{(a + 1)^2}{a(a + 2)}.$$

One easily verifies that $F_a$ has a continuous derivative on $[0, 1/2]$. The function $F$, defined by

$$F(z) = \sum_{a=1}^{\infty} F_a(z), \quad 0 \leq z \leq \frac{1}{2},$$

is the distribution function of the sequence $|d_n - d_{n+1}|$. It can be written in the closed form given in the statement of the theorem. \qed

**Theorem 4.** For almost all $x$ one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |d_n - d_{n+1}| = \frac{1 - \gamma}{\log 2} - \frac{1}{2} = 0.10994 \ldots,$$

where $\gamma$ is Euler’s constant.

**Proof.** Denote the first moment of $F_a$ by $M_a$. Then

$$M_a = \int_{0}^{1} z F'_a(z) \, dz = \frac{1}{\log 2} \left( \frac{1}{2} \log \frac{a + 2}{a} - \frac{1}{a + 1} \right).$$

Summing over $a$ yields

$$\lim_{A \to \infty} \sum_{a=1}^{A} M_a = \frac{1}{\log 2} \lim_{A \to \infty} \left( \frac{1}{2} \log \frac{1}{2} (A + 1)(A + 2) - \sum_{a=2}^{A+1} \frac{1}{a} \right).$$

In this we substitute the classical

$$\sum_{a=2}^{A+1} \frac{1}{a} = \log(A + 1) + \gamma - 1 + o(1), \quad A \to \infty,$$

and obtain, for almost all $x$, the expectation of $|d_n - d_{n+1}|$ as given by the theorem. \qed
3. The distribution of the sequence \( d_n + d_{n+1} \)

We introduce the following intervals:

\[
\Delta_m^{(0)} = \left[ \frac{2m + 1}{m + 1}, \frac{4m + 4}{2m + 3} \right], \quad m = 1, 2, \ldots
\] (8)

and

\[
\Delta_m^{(1)} = \left[ \frac{4m + 4}{2m + 3}, \frac{2m + 3}{m + 2} \right], \quad m = 0, 1, 2, \ldots
\] (9)

These intervals are pairwise disjoint. Their natural order is

\[
\Delta_0^{(1)}, \Delta_1^{(0)}, \Delta_1^{(1)}, \Delta_2^{(0)}, \Delta_2^{(1)}, \Delta_3^{(0)}, \Delta_3^{(1)}, \ldots
\]

and together they fill up the interval \( \left[ \frac{4}{3}, 2 \right) \).

Put

\[
\Delta_m^{(0)} = \bigcup_{m=1}^{\infty} \Delta_m^{(0)} \quad \text{and} \quad \Delta_m^{(1)} = \bigcup_{m=0}^{\infty} \Delta_m^{(1)}.
\]

The sets \( \Delta^{(0)} \) and \( \Delta^{(1)} \) have Lebesgue measure

\[
\frac{5}{3} - 2 \log 2 = 0.28037 \ldots \quad \text{and} \quad 2 \log 2 - 1 = 0.38629 \ldots,
\]

respectively.

Next we define for every \( z \in \left[ \frac{4}{3}, 2 \right) \) two integers \( m \) and \( n \) by

\[
m = m(z) = \left\lfloor \frac{z - 1}{2 - z} \right\rfloor \quad \text{and} \quad n = n(z) = \left\lfloor \frac{2z - 2}{2 - z} \right\rfloor.
\]

Clearly, \( n = 2m \) or \( n = 2m + 1 \). Write \( \lambda(z) = n - 2m \). The intervals from (8) and (9) are constructed in such a way that \( \lambda \) is the characteristic function of \( \Delta^{(1)} \).

Finally, we use the abbreviation

\[
h(z) = h(m, n) = \sum_{k=m+1}^{n} \frac{1}{k}.
\]

**Theorem 5.** Let \( F \) be the function defined on \( \left[ \frac{4}{3}, 2 \right] \) by

\[
F(z) = \frac{1}{\log 2} \left( 2h(z)(z - 1) + \lambda(z)(z - \log(z - 1) - 2)
\right.
\]

\[
+ \log(m + 1) - \log(n + 1 + \lambda(z)) \bigg), \quad \frac{4}{3} \leq z < 2,
\]

\[
F(2) = 1.
\]

This function \( F \) is monotonically increasing from 0 in \( z = \frac{4}{3} \) to 1 in \( z = 2 \) and has a continuous derivative.
It is for almost all \( x \) the distribution function of the sequence
\[
d_n + d_{n+1}, \quad n = 1, 2, \ldots
\]

**Remark.** On each interval \( \Delta_m^{(0)} \) from (8), \( F \) is the simple linear function
\[
\frac{1}{\log 2} \left( 2h(m, 2m)(z - 1) + \log \frac{m + 1}{2m + 1} \right).
\]

For example,
\[
F(z) = \frac{1}{\log 2} \left( z - 1 + \log 2 - \log 3 \right), \quad z \in \Delta_1^{(0)} = \left[ \frac{3}{2}, \frac{8}{5} \right],
\]
\[
F(z) = \frac{1}{\log 2} \left( \frac{7}{6}(z - 1) + \log 3 - \log 5 \right), \quad z \in \Delta_2^{(0)} = \left[ \frac{5}{3}, \frac{12}{7} \right].
\]

On an interval from (9) there are only minor changes from this. The coefficient \( h(m, 2m) \) is replaced by \( h(m, 2m + 1) \) which means that a term \( 1/(2m + 1) \) is added to it; the term \( \log((m + 1)/(2m + 1)) \) changes into \( \log((m + 1)/(2m + 3)) \) and finally the term
\[
\frac{1}{\log 2} (z - \log(z - 1) - 2),
\]
which is asymptotically \( \frac{1}{2\log 2}(z - 2)^2 \) for \( z \to 2 \), is added.

For example,
\[
F(z) = \frac{1}{\log 2} \left( 3z - \log(z - 1) - 4 - \log 3 \right), \quad z \in \Delta_0^{(1)} = \left[ \frac{4}{3}, \frac{3}{2} \right],
\]
\[
F(z) = \frac{1}{\log 2} \left( \frac{8}{3}z - \log(z - 1) - \frac{11}{3} + \log 2 - \log 5 \right), \quad z \in \Delta_1^{(1)} = \left[ \frac{8}{5}, \frac{5}{3} \right].
\]

On an interval \( \Delta_m^{(0)} \) from (8) one has
\[
F'(z) = \frac{1}{\log 2} 2h(z)
\]
and on an interval \( \Delta_m^{(1)} \) from (9)
\[
F'(z) = \frac{1}{\log 2} \left( 2h(z) + 1 - \frac{1}{z - 1} \right).
\]
Figure 2. Distribution function of $d_n + d_{n+1}$.

From this and from

$$\lim_{z \uparrow 2} h(z) = \log 2$$

it follows that

$$\lim_{z \uparrow 2} F'(z) = 2.$$

Further,

$$\lim_{z \downarrow \frac{5}{3}} F'(z) = 0.$$

Between $5/3$ and $2$, the graph of $F$ is on the scale of Figure 2 not distinguishable from a line segment.

Proof. Denote by $R_a(z)$ that part of $R_a$ which lies under the line $\alpha + \beta = z$. Then

$$\lim_{N \to \infty} \frac{1}{N} \# \{ n; n \leq N, d_n + d_{n+1} < z, a_{n+2} = a \} = \nu(R_a(z)).$$

Call this expression $F_a(z)$ and consider it as a function on the interval $[\frac{4}{3}, 2]$. Divide this interval into three subintervals, $I_a^{(0)}$, $I_a^{(1)}$ and $I_a^{(2)}$ by the two points
(2a + 2)/(a + 2) and (2a + 1)/(a + 1). Some elementary integration leads to

\[ F_a(z) = \begin{cases} 
0, & z \in I_a^{(0)}, \\
\frac{1}{\log 2} \left( \frac{a + 2}{a} (z - 1) - \log(z - 1) + \log \frac{a}{a + 2} - 1 \right), & z \in I_a^{(1)}, \\
\frac{1}{\log 2} \left( -z + \log(z - 1) + 2 + \log \frac{(a + 1)^2}{a(a + 2)} \right), & z \in I_a^{(2)}. 
\end{cases} \]

To get the expression for \( F_a \) on \( I_a^{(1)} \) one has to integrate over two separate regions, one to the left and one to the right of the line \( \alpha = \frac{(a + 1)}{(a + 2)} \).

The easiest way to calculate the form of \( F_a \) on \( I_a^{(2)} \) is to integrate over the complement of \( R_a(z) \) and to subtract this from \( \nu(R_a) \), which equals, as is well-known,

\[ \frac{1}{\log 2} \log \frac{(a + 1)^2}{a(a + 2)}. \]

To obtain the required distribution function we must sum over \( a \). We take a \( z \) and observe that

\[ z \in I_{m+1}^{(1)}, I_{m+2}^{(1)}, I_{m+3}^{(1)}, \ldots, I_n^{(1)} \]

and that \( z \) is not contained in any other interval \( I_a^{(1)} \).

Hence

\[ F(z) = \frac{1}{\log 2} \sum_{a=1}^{m} \left( -z + \log(z - 1) + 2 + \log \frac{(a + 1)^2}{a(a + 2)} \right) + \frac{1}{\log 2} \sum_{a=m+1}^{n} \left( \frac{a + 2}{a} (z - 1) - \log(z - 1) + \log \frac{a}{a + 2} - 1 \right), \]

and after some rearrangements one obtains the form as given in the statement of the theorem. □

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