Uniform Distribution Theory **7** (2012), no.1, 71–73

uniform distribution theory

A FOOTNOTE TO *THE LEAST NON ZERO DIGIT OF* $n!$ *IN BASE* 12

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ABSTRACT. We continue the work initiated with Imre Ruzsa, showing that for any $a \in \{3, 6, 9\}$, there exist infinitely many integers n such that the least non zero digit of $n!$ in base twelve is equal to a.

Communicated by Pierre Liardet

Dedicated to the memory of Gérard Rauzy

The least (also called "last" or "final") non zero digit of $n!$ in a given base b, denoted by $\ell_b(n!)$, has attracted the attention of several autors: numerical values can be found on N. J. A. Sloanes's project [6]; S. Kakutani [5] seems to be the first one to show the 5-automaticity of $\ell_{10}(n!)$. M. Dekking [2] studied the cases $b = 3$ and $b = 10$; more recently G. Dresden [4] gave a detailed study of the case $b = 10$.

Let us explain why I. Ruzsa and I studied in [3] the case $b = 12$. A-M. Legendre has shown that the p-valuation of $n!$, denoted by $v_p(n!)$, is equal to $(n - s_p(n))/(p - 1)$ where p denotes a prime number and $s_p(n)$ denotes the sum of the digits of n written in base p; this implies that the number of zeroes of $n!$ is

$$\min_i \left\lfloor \frac{n - s_{p_i}(n)}{a_i(p_i - 1)} \right\rfloor, \text{ when } b = p_1^{a_1} p_2^{a_2} \dots \text{ with } a_1(p_1 - 1) \geq a_2(p_2 - 1) \geq \dots$$

If b is a prime power or if $a_1(p_1 - 1) > a_2(p_2 - 1)$, then the minimum is $\lfloor (n - s_{p_1}(n))/a_1(p_1 - 1) \rfloor$ when n is sufficiently large, and the sequence $(\ell_b(n!))_n$ is p_1-automatic. The smallest value of b for which $a_1(p_1 - 1) = a_2(p_2 - 1)$ occurs

2010 Mathematics Subject Classification: 11A63, 11B85.
Keywords: significant digits, factorial, base 12.
Supported by Université de Bordeaux (IPB) and CNRS (UMR 5465).

when $b = 12$ and in this case, we suspect that the sequence $(\ell_b(n!))_n$ is not automatic; however I. Ruzsa and I showed that this sequence coincides on a set of density 1 with a 3-automatic sequence and we proved the following

Theorem 1. *Let $1 \le a \le 11$. The sequence $\{n : \ell_{12}(n!) = a\}$ has an asymptotic density, which is $1/2$ if $a = 4$ or $a = 8$ and 0 otherwise.*

Thus 4 and 8 belong to the range of $n \mapsto \ell_{12}(n!)$. It is easy to check (thanks to Pari, or by consulting the entry `A136698` of [6]) that the range of this map is indeed the whole set $\{a : 1 \le a \le 11\}$: for example, the values $1, 2, 3, 5, 6, 7, 9, 10$, and 11 are respectively attained when $n = 46, 23, 30, 19, 28, 21, 31, 22$, and 18. Partially answering a question raised in [3], we show the following

Theorem 2. *The least significant digit $\ell_{12}(n!)$ of $n!$ in base 12 takes infinitely often each of the values $3, 6$ and 9.*

Theorem 2 will be deduced from the following

Proposition 3. *Let n be divisible by 144 and be such that $v_3(n!) \ge v_4(n!) + 2$. Then for any $a \in \{3, 6, 9\}$ there exists $k \in \{0, 2, 3, 7\}$ such that $\ell_{12}((n+k)!) = a$.*

Proof. Assume first that $v_3(n!) \ge v_4(n!) + 2$. In base 12, let us write $n! = \overline{\cdots m(n!)\ell(n!)000\cdots 00}$, with $\ell(n!) = \ell_{12}(n!) \ne 0$, then the number $\overline{m(n!)\ell(n!)}$ is divisible by 9, and so $\ell(n!) \in \{3, 6, 9\}$.
Let us further remark that if $\ell(n!) = 6$, then 9 divides $12m(n!) + 6$, and so 3 divides $2m(n!)+1$; since $2m(n!)+1$ is odd, it is either congruent to 3 or 9 modulo 12, and then $6m(n!) + 3$ is respectively congruent either to 9 or 3 modulo 12.

Assume now that n is divisible by 144. One readily checks that $(n+1)(n+2)$ is congruent to 2 modulo 144, that $(n+1)(n+2)(n+3)$ is congruent to 6 modulo 144 and that $(n + 1)\cdots(n + 7)/144$ is congruent to -1 modulo 12.

Combining the above results, if we assume that n is divisible by 144 and is such that $v_3(n!) \ge v_4(n!) + 2$, we have the following

(1) If $\ell(n!) = 3$, then $\ell((n + 2)!) = 6$ and $\ell((n + 7)!) = 9$,
(2) If $\ell(n!) = 9$, then $\ell((n + 2)!) = 6$ and $\ell((n + 7)!) = 3$,
(3) If $\ell(n!) = 6$, then $\ell((n+2)!)$ is either 3 or 9 and $\ell((n+3)!)$ is respectively either 9 or 3.

This proves Proposition 3. □

Proof of Theorem 2. By Legendre's above-mentioned theorem (cf. [1], Cor. 3.2.2 or [3], Lemma 1), we have $v_3(n!) = (n - s_3(n))/2$ and $v_4(n!) = \lfloor (n - s_2(n))/2 \rfloor$. For $m \ge 0$, let $n = 16 \cdot 3^{16384m+9}$. By Euler's theorem, we know that $3^{16384} =$

$3^{2^{14}}$ is congruent to 1 modulo $32768 = 2^{15}$; we further have $3^9 = 19683 = 2^{14} + 2^{11} + 2^{10} + 2^7 + 2^6 + 2^5 + 2^1 + 2^0$, and so, for $m \geq 0$ we have $s_2(n) \geq 8$ (and indeed $s_2(n) \geq 9$ for $m \geq 1$, a remark that we do not use). We thus have $v_4(n!) = \lfloor (n - s_2(n))/2 \rfloor \leq (n-8)/2$. On the other hand, we have $s_3(n) = s_3(16) = s_3(9{+}2{\times}3{+}1) = 4$, and $v_3(n!) = (n-4)/2 = (n-8)/2+2 \geq v_4(n!)+2$. Since $n = 16 \cdot 3^{16384m+9}$ is divisible by 144, Proposition 3 implies that for any $m \geq 0$ we have $\{3, 6, 9\} \subset \{\ell_{12}((n+k)!) \,/\, k = 0, 2, 3, 7\}$, which proves Theorem 2.

Let me conclude with two open questions: do all the numbers from $\{1, 2, \ldots, 10, 11\}$ occur infinitely often in the sequence $(\ell_{12}(n!))_n$? How often do $3, 6$ and 9 occur in $(\ell_{12}(n!))_{n \leq N}$ when N is large?

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Received September 3, 2011
Accepted September 28, 2011

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