ON THE LIMIT DISTRIBUTION OF CONSECUTIVE ELEMENTS OF THE VAN DER CORPUT SEQUENCE

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ABSTRACT. Recently, Fialová and Strauch, Uniform Distribution Theory, 6(1):101-125, 2011, calculated the asymptotic distribution function (adf) of the two-dimensional sequence \((\phi_b(n), \phi_b(n+1))_{n \geq 0}\), where \((\phi_b(n))_{n \geq 0}\) denotes the van der Corput sequence in base \(b\). In the present paper we solve the general problem asking for the limit distribution of \((\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n \geq 0}\). We use the fact that the van der Corput sequence can be seen as the orbit of the origin under the ergodic von Neumann-Kakutani transformation.

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1. Introduction

In the open problem collection on the web site of Uniform distribution theory the following problem is stated:

Let \((\phi_b(n))_{n \geq 0}\) denote the van der Corput sequence in base \(b\). Find the distribution of the sequence \((\phi_b(n), \phi_b(n+1), \ldots, \phi_b(n+s-1))_{n \geq 0}\) in \([0, 1)\).

The case \(s = 2\) has recently been solved by Fialová and Strauch \[3\]. They showed that every point \((\phi_b(n), \phi_b(n+1))_{n \geq 0}\) lies on the line segment

\[y = x - 1 + \frac{1}{b^k} + \frac{1}{b^k+1}, \quad x \in \left[1 - \frac{1}{b^k}, 1 - \frac{1}{b^k+1}\right]\]
for $k \geq 0$. Furthermore they could give an explicit formula for the asymptotic
distribution function $g(x, y)$ of $(\phi_b(n), \phi_b(n + 1))_{n \geq 0}$ to calculate the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\phi_b(n) - \phi_b(n + 1)| = \int_{0}^{1} \int_{0}^{1} |x - y| dg(x, y) = \frac{2(b-1)}{b^2}
\]
previously demonstrated by Pillichshammer and Steinerberger [13]. They also
noted that the adf of $(\phi_b(n), \phi_b(n + 1))_{n \geq 0}$ is a copula.

In this article we solve the problem for the sequence $(\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0}$ for $s > 2$. A multi-dimensional extension of the van der Corput sequence $(\phi_b(n))_{n \geq 0}$, is given by the so-called Halton sequence,
$(\phi_{b_1}(n), \phi_{b_2}(n), \ldots, \phi_{b_s}(n))_{n \geq 0}$ which is uniformly distributed if and if the bases $b_1 \leq i \leq s$ are co-prime (see [7]). These sequences are well-studied objects in
discrepancy theory, since they belong to the class of so-called low discrepancy
sequences. For classical results in discrepancy theory, on low discrepancy
sequences and the van der Corput sequence see e.g. [1], [2] or [9].

Recently, several authors investigated the ergodic properties of low discrep-
ancy sequences, see e.g. [6] and [8]. In the case of van der Corput sequences this
can be done using the so-called von Neumann-Kakutani transformation, which
will be discussed in the second section.

The outline of this article is as follows: in the second section we define the van
der Corput sequence and the von Neumann-Kakutani transformation and re-
call their basic properties. In the third section we state our main results on the
distribution of $(\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0}$.

2. van der Corput sequence and von Neumann-Kakutani
transformation

Let $b \in \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then for every $n \in \mathbb{N}_0$, we can write
\[
n = \sum_{i \geq 0} n_i b^i
\]
where $n_i \in \{0, 1, \ldots, b-1\}, i \geq 0$. The above sum is called $b$-adic representation
of $n$. The $n_i$ are uniquely determined and at most a finite number of $n_i$ are
non-zero. Furthermore, every real $x \in [0, 1)$ has a $b$-adic representation of the
following form
\[
x = \sum_{i \geq 0} x_i b^{-i-1}
\]
where \( x_i \in \{0, 1, \ldots, b-1\}, i \geq 0 \). We call \( x \) a \( b \)-adic rational if \( x = ab^{-c} \), where \( a \) and \( c \) are positive integers and \( 0 \leq a < b^c \). For all \( b \)-adic integers there are exactly two representations of the form \((\Pi)\), one where \( x_i = 0, i \geq i_0 \) and one where \( x_i = b - 1, i \geq i_0 \) for sufficiently large \( i_0 \in \mathbb{N} \). If we restrict ourselves to representations with \( x_i \neq b - 1 \) for infinitely many \( i \), then the coefficients \( x_i \) in \((\Pi)\) are uniquely determined for all \( x \in [0, 1) \).

For \( n \in \mathbb{N}_0 \) we define the so-called radical-inverse function or Monna map \( \phi_b(n): \mathbb{N}_0 \rightarrow [0, 1) \) by

\[
\phi_b(n) = \phi_b \left( \sum_{i \geq 0} n_i b^i \right) := \sum_{i \geq 0} n_i b^{-i-1}.
\]

Note that \( \phi_b(n) \) maps \( \mathbb{N}_0 \) to the set of \( b \)-adic rationals in \([0, 1)\), and therefore the image of \( \mathbb{N}_0 \) under \( \phi_b(n) \) is dense in \([0, 1)\).

**Definition 2.1.** The van der Corput sequence in base \( b \) is defined as \((\phi_b(n))_{n \geq 0}\).

It is a classical result that the van der Corput sequence is uniformly distributed in \([0, 1)\), see e.g. [9]. Furthermore, its \( s \)-dimensional extension, the Halton sequence given by \((\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n \geq 0}\) for co-prime bases \( b_i, 1 \leq i \leq s \), is uniformly distributed on \([0, 1)^s\). Properties of the van der Corput and the Halton sequence are very well-understood, since they are so-called low discrepancy sequences, which are central objects in Quasi-Monte Carlo integration.

A second approach to define the van der Corput sequence is by using the von Neumann-Kakutani transformation \( T_b: [0, 1) \rightarrow [0, 1) \). For any integer \( b \geq 2 \) the inductive construction of \( T_b \) is as follows: at first \([0, 1)\) is split into \( b \) intervals \( I_1^1 = \left[ \frac{1}{b}, \frac{i+1}{b} \right) \) for \( i = 0, 1, \ldots, b-1 \). Then the transformation \( T_{1,b}: \left[ 0, \frac{b-1}{b} \right) \rightarrow \left[ \frac{1}{b}, 1 \right) \) is defined as translation of \( I_1^1 \) into \( I_{i+1}^1 \) for \( i = 0, 1, \ldots, b-1 \). The next step is to divide all intervals \( I_i^1 \) into \( b \) subintervals of the form \( I_i^2 = \left[ \frac{i}{b^2}, \frac{i+1}{b^2} \right) \) for \( i = 0, 1, \ldots, b^2-1 \). Transformation \( T_{2,b}: \left[ 0, \frac{b^2-1}{b^2} \right) \rightarrow \left[ \frac{1}{b^2}, 1 \right) \) is given as the extension of \( T_{1,b} \) which translates \( I_{b^2-b+1}^2 \) into \( I_{b^2-b+i+1}^2 \) for \( i = 0, 1, \ldots, b-1 \). Such a construction is called splitting-and-stacking-construction and is illustrated in Figure 2 for \( b = 2 \). Finally we define the von Neumann-Kakutani transformation as \( T_b = \lim_{n \rightarrow \infty} T_{n,b} \). A plot of the transformation \( T_2 \) is given in Figure [2]. By an observation of Lambert [10], [11] (see also Hellekalek [7]) the van der Corput sequence in base \( b \) is exactly the orbit of the origin under \( T_b \), which means that

\[
(T_b^n 0)_{n \geq 0} = (\phi_b(n))_{n \geq 0}, \quad b \geq 2,
\]

where \( T_b^n x \) denotes the value of \( x \) under after \( n \) iterations of \( T_b \).

For a proof of the ergodicity and measure-preserving properties of the von
Figure 1. The first two steps of a splitting-and-stacking-construction in base $b = 2$.

Figure 2. The von Neumann-Kakutani transformation in base $b = 2$.

Neumann-Kakutani transformation, see e.g. [4] or [5]. It follows from the ergodicity of the von Neumann-Kakutani transformation that $(T_b^n x)_{n \geq 0}$ is uniformly distributed for almost every $x \in [0, 1)$. Furthermore, it can be shown that the von Neumann-Kakutani transformation is uniquely ergodic, which implies that
(T^n_b x)_{n \geq 0} is uniformly distributed for every x ∈ [0, 1), see e.g. [6]. Moreover, Pagès [12] showed that the orbit of the von Neumann-Kakutani transformation starting at an arbitrary point x ∈ [0, 1) is a low discrepancy sequence. Another possible generalization of the van der Corput sequence is the so-called randomized van der Corput sequence (T^n_b X)_{n \geq 0} where X is uniformly distributed on [0, 1), see [14].

Recently, Fialová and Strauch solved the problem of calculating the limit distribution of the sequence (φ_b(n), φ_b(n + 1))_{n \geq 0}. They also concluded that the limit distribution is a copula. We consider the multi-dimensional extension of this problem. By (2)

(φ_b(n), φ_b(n + 1))_{n \geq 0} = (T^n_b 0, T^{n+1}_b 0)_{n \geq 0} = (T^n_b 0, T_b(T^n_b 0))_{n \geq 0}.

By the fact that (T^n_b 0)_{n \geq 0} is uniformly distributed on [0, 1) one can show that (φ_b(n), φ_b(n + 1))_{n \geq 0} is uniformly distributed on

Γ = \{(x, y) : y = T_b x\}.

Note that Γ coincides with the graph of the von Neumann-Kakutani transformation in Figure 2. In the next section we use this approach to find the limit distribution of (φ_b(n), φ_b(n + 1), ..., φ_b(n + s − 1))_{n \geq 0} for arbitrary s ≥ 2.

3. The limit distribution of consecutive elements of the van der Corput sequence

In the sequel we assume that b, s are fixed. Let T denote the von Neumann-Kakutani transformation in base b as described in Section 2. We define a map γ(t) : [0, 1) → [0, 1]^s by setting

γ(t) := \begin{pmatrix} t \\ Tt \\ T^2t \\ \vdots \\ T^{s-1}t \end{pmatrix}

and

Γ := \{(x_1, x_2, \ldots, x_s) ∈ [0, 1]^s : x_i = T^{i-1} x_1, i = 2, \ldots, s\} = \{γ(t) : t ∈ [0, 1)\}.

The Lebesgue measure λ_1 on [0, 1) induces a measure ν on Γ by setting

ν(A) = λ_1(\{t : γ(t) \in A\}), \quad A ⊂ Γ.
Furthermore, $\nu$ induces a measure $\mu$ on $[0, 1)^s$ by embedding $\Gamma$ into $[0, 1)^s$. More precisely for every measurable subset $B \subseteq [0, 1)^s$ we set

$$\mu(B) = \nu(B \cap \Gamma).$$

**Theorem 3.1.** The limit measure of $(\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0}$ is $\mu$.

**Proof.** As mentioned in Section 2, we can rewrite $(\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0} = (T^n0, T^{n+1}0, \ldots, T^{n+s-1}0)_{n \geq 0} = (T^n0, T(T^n0), \ldots, T^{s-1}(T^n0))_{n \geq 0}$.

Since $(T^n0)_{n \geq 0}$ is uniformly distributed on $[0, 1)$ and $T$ is a measure-preserving transformation with respect to $\lambda_1$, it follows immediately that $(T^i(T^n0))_{n \geq 0}$ is uniformly distributed on $[0, 1)$ for $i = 1, \ldots, s - 1$. Moreover, by construction $(T^n0, T(T^n0), \ldots, T^{s-1}(T^n0))_{n \geq 0} \in \Gamma$ for all $n \geq 0$.

Now consider a Jordan measurable set $B \in [0, 1)^s$. We define the empirical measure of the first $N$ points of $(T^n0, \ldots, T^{s-1}(T^n0))_{n \geq 0}$ as

$$\mu_N(B) = \frac{1}{N} \# \{0 \leq n \leq N : (T^n0, T(T^n0), \ldots, T^{s-1}(T^n0)) \in B\}.$$ 

We have

$$\lim_{N \to \infty} \mu_N(B) = \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n \leq N : (T^n0, T(T^n0), \ldots, T^{s-1}(T^n0)) \in B\}$$

$$= \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n \leq N : (T^n0, T(T^n0), \ldots, T^{s-1}(T^n0)) \in B \cap \Gamma\}$$

$$= \lim_{N \to \infty} \frac{1}{N} \# \{0 \leq n \leq N : T^n0 \in \text{Projection}_{x_1}(B \cap \Gamma)\}$$

$$= \lambda_1(\text{Projection}_{x_1}(B \cap \Gamma))$$
\begin{align*}
\nu(B \cap \Gamma) &= \mu(B)\\
&= \nu(B) \cap \Gamma = \mu(B)
\end{align*}

where the fourth equation holds since \((T^n 0)_{n \geq 0}\) is uniformly distributed on \([0, 1]\) and since the map \(t \rightarrow Tt\) is a bijection, and where Projection\(_{x_1}(A)\) denotes the projection of \(A\) onto its first coordinate.

\begin{remark}
Note that the measure \(\mu\) is a copula on \([0, 1]^s\) for every \(s\) since every distribution function of a multi-dimensional sequence \((x^1_n, \ldots, x^n_s)_{n \geq 0}\) is a copula if the sequences \((x^1_n)_{n \geq 0}, \ldots, (x^n_s)_{n \geq 0}\) are uniformly distributed on \([0, 1]\).
\end{remark}

\begin{remark}
The set \(\Gamma\) is a collection of countably many line segments in \([0, 1)^s\), which are parallel to the diagonal. Informally speaking Theorem 3.1 means that \((\phi_b(n), \phi_b(n + 1), \ldots, \phi_b(n + s - 1))_{n \geq 0}\) is uniformly distributed on \(\Gamma\).
\end{remark}

\begin{remark}
By the unique ergodicity of \(T\), the conclusion of Theorem 3.1 holds also for the sequence \((T^n x, T(T^n x), \ldots, T^{s-1}(T^n x))_{n \geq 0}\) for arbitrary \(x \in [0, 1]\).
\end{remark}

\begin{remark}
Another class of uniformly distributed sequences which can be seen as the orbits of certain points under an ergodic transformation are sequences of the form \(\{n \alpha\}_{n \geq 0}\), where \(\{x\}\) denotes the fractional part of \(x\) and \(\alpha\) is irrational. In this case the corresponding transformation \(\hat{T}\) is simply the rotation \(\hat{T}: x \mapsto x + \alpha \mod 1\). It can easily be shown that the limit distribution of consecutive elements \((\{n \alpha\}, \{(n+1) \alpha\}, \ldots, \{(n+s-1) \alpha\})_{n \geq 0}\) is the uniform distribution on the curve \(\hat{\Gamma}\) which is given by
\[
\hat{\Gamma} := \{(t, \hat{T}t, \ldots, \hat{T}^{s-1}t), t \in [0, 1)\}.
\]

However, since in this case the transformation \(\hat{T}\) has a particularly simple structure, the same result can also be easily obtained using analytic arguments.

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