Uniform Distribution Theory **9** (2014), no.1, 21–25

uniform distribution theory

ON THE FIRST DIGITS OF THE FIBONACCI NUMBERS AND THEIR EULER FUNCTION

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ABSTRACT. Here, we show that given any two finite strings of base b digits, say s_1 and s_2, there are infinitely many Fibonacci numbers F_n such that the base b representation of F_n starts with s_1 and the base b representation of $\phi(F_n)$ starts with s_2.

Communicated by Oto Strauch

1. Introduction

Let $b \geq 2$ be and integer. Let $s_1 = \overline{c_1 \cdots c_k}_{(b)}$ be a positive integer s_1 written in base b. Washington [4] proved that there exist infinitely many Fibonacci numbers F_n whose base b representation starts with s_1. In fact, the first digits of the Fibonacci sequence obey Benford's law in that the proportion of the positive integers n such that F_n starts with s_1 is precisely $\log((s_1+1)/s_1)/\log b$. Here, we take this one step further. Let $\phi(m)$ be the Euler function of the positive integer m. We put $s_2 = \overline{d_1 \ldots d_\ell}_{(b)}$ for some other positive integer written in base b and prove the following theorem.

Theorem. *Given positive integers $s_1 = \overline{c_1 \cdots c_k}_{(b)}$ and $s_2 = \overline{d_1 \ldots d_\ell}_{(b)}$ written in base b, there exist infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\phi(F_n)$ starts with the digits of s_2.*

We use the fact that with $(\alpha, \beta) = \big((1+\sqrt{5})/2, (1-\sqrt{5})/2\big)$ the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{holds for all} \qquad n \geq 0.$$

For a positive real number x we write $\log x$ for the natural logarithm of x, and $\lfloor x \rfloor$, respectively, $\{x\}$, for the integer part, respectively, fractional part of x.

2010 Mathematics Subject Classification: 11B39,11K36.
Keywords: Euler function, Fibonacci numbers, digits distribution.

2. The proof

By replacing s_1 with $s_1 b^m$ for some positive integer m, if needed, whose effect is adding m zeros at the end of the base b representation of s_1, we may assume that $s_1 > s_2$. By replacing s_1, s_2 by $s_1 b^m$, respectively, $s_2 b^m$ for an arbitrary positive integer m, we may assume that the length of the base b representation of s_1, that is k, is as large as we wish. In Section 4 of [2], it is shown that $\phi(F_n)/F_n$ is dense in $[0,1]$. So, we take $\varepsilon \in \big(0, 1/(15b^{2k})\big)$ and choose a positive integer a such that

$$\frac{\phi(F_a)}{F_a} \in \left(\frac{s_2}{s_1} + \varepsilon, \frac{s_2}{s_1} + 2\varepsilon\right). \tag{1}$$

Now we take any prime $p > F_a$ and look at F_{ap}. Since $p > F_a$, it follows that

$$F_{ap} = F_a \left(\frac{F_{ap}}{F_a}\right),$$

and the two factors F_a and F_{ap}/F_a on the right above are coprime (indeed, the only common prime factor of these two numbers could be p, which is not the case since $p > F_a$). Any prime factor q of F_{ap}/F_a is a *primitive prime factor* of F_{dp} for some divisor d of a. Recall that a prime number q is said to be a primitive prime factor of F_n if q divides F_n, but does not divide any F_m for $1 \le m < n$. One of the properties of primitive prime factors q of F_n when $n > 5$ is that $q \equiv \pm 1 \pmod{n}$. In particular, every prime factor q of F_{ap}/F_a is congruent to $\pm 1 \pmod{p}$.

Let $q_1, \ldots, q_t$ be all the prime factors of F_{ap}/F_a. Then

$$(2p-1)^t \le q_1 \cdots q_t \le \frac{F_{ap}}{F_a} \le F_{ap} \le \alpha^{ap}.$$

Thus, $t = O(p/\log p)$. Then

$$\begin{aligned}
\frac{\phi(F_{ap})}{F_{ap}} &= \left(\frac{\phi(F_a)}{F_a}\right) \prod_{i=1}^{t} \left(1 - \frac{1}{q_i}\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(-\sum_{i=1}^{t} \frac{1}{q_i} + O\left(\sum_{q \ge q_1} \frac{1}{q^2}\right)\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(O\left(\frac{t}{q_1}\right)\right) \\
&= \frac{\phi(F_a)}{F_a} \exp\left(O\left(\frac{1}{\log p}\right)\right)
\end{aligned}$$

$$= \quad \frac{\phi(F_a)}{F_a}\left(1+O\left(\frac{1}{\log p}\right)\right).$$

It implies that, if $p > \exp(\kappa\varepsilon^{-1})$, where $\kappa > 0$ is some absolute constant, then

$$\frac{\phi(F_{ap})}{F_{ap}} \in \left(\frac{s_2}{s_1}+0.5\varepsilon, \frac{s_2}{s_1}+1.5\varepsilon\right). \tag{2}$$

We now follow Washington's argument [4] to prove that there exist infinitely many primes p such that the base b representation of F_{ap} starts with s_1. For this, it is enough to show that

$$F_{ap} = s_1 b^N + \zeta_{ap} \qquad \text{for some integer} \qquad 0 \le \zeta_{ap} \le b^N - 1. \tag{3}$$

Note that since $q_1 \ge 2p-1$, it follows that if p is sufficiently large (say, $p > b^k$), then F_{ap} cannot equal $s_1 b^N$, and in particular, if in the above formula (3) we have $\zeta_{ap} \ge 0$, then in fact $\zeta_{ap} \ge 1$. The above formula (3) yields

$$\alpha^{ap} = \sqrt{5}s_1 b^N + \sqrt{5}\zeta_{ap} + \beta^{ap} = \sqrt{5}s_1 b^N\left(1+x_{ap}\right).$$

Since $\zeta_{ap} \ge 1$, it follows that

$$\sqrt{5}\zeta_{ap} + \beta^{ap} > \sqrt{5}-1 > 1, \quad \text{and} \quad 0 < x_{ap} = \frac{\sqrt{5}\zeta_{ap}+\beta^{ap}}{\sqrt{5}s_1 b^N}.$$

So, if $x_{ap} \in (0, 1/b^k)$, and $p > b^k$ is sufficiently large, it then follows that $\zeta_{ap} < b^N$, which is what we want. Thus,

$$ap\log\alpha = \log(\sqrt{5}s_1) + N\log b + \log(1+x_{ap}),$$

or

$$ap\frac{\log\alpha}{\log b} - N - \left\lfloor \frac{\log(\sqrt{5}s_1)}{\log b}\right\rfloor = \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\} + \frac{\log(1+x_{ap})}{\log b}. \tag{4}$$

Observe that $\log(\sqrt{5}s_1)/\log b$ is never an integer. Assume that k is sufficiently large such that

$$\frac{1}{b^k\log b} < 1 - \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\}.$$

Then putting

$$\delta = \frac{\log(1+1/b^k)}{\log b},$$

we see that a relation like (4) with $x_{ap} \in (0, 1/b^k)$ holds provided that

$$\left\{p\left(\frac{a\log\alpha}{\log b}\right)\right\} \in \left(\left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\}, \left\{\frac{\log(\sqrt{5}s_1)}{\log b}\right\} + \delta\right). \tag{5}$$

The number $\gamma = a \log \alpha / \log b$ is irrational. By a result of Vinogradov [3], the sequence of fractional parts $\{p\gamma\}_{p \text{ prime}}$ is uniformly distributed. In particular, containment (5) holds for a positive proportion of primes p, and therefore certainly for infinitely many of them. So, indeed relation (3) holds. Relation (2) now shows that

$$\phi(F_{ap}) = s_2 b^N + \theta,$$

where

$$\theta \in \left(\zeta_{ap}\left(\frac{s_2}{s_1} + 0.5\varepsilon\right) + 0.5\varepsilon b^N, \zeta_{ap}\left(\frac{s_2}{s_1} + 1.5\varepsilon\right) + 1.5\varepsilon s_1 b^N\right).$$

Since $\varepsilon < 1/(15b^{2k})$, the above upper bound is

$$\begin{aligned} \zeta_{ap}\left(\frac{s_2}{s_1} + 1.5\varepsilon\right) + 1.5\varepsilon s_1 b^N &< (b^N - 1)\left(\frac{b^k - 1}{b^k} + \frac{0.1}{b^k}\right) + \frac{0.1(b^k - 1)}{b^{2k}} b^N \\ &< b^N - 1, \end{aligned}$$

where the last inequality above is implied by

$$\frac{1}{9} < \frac{b^N - 1}{b^N},$$

which holds true for all $b \geq 2$ and $N \geq 1$. This completes the proof of the theorem.

3. Comments

It was shown in [1] that with $\sigma(m)$ being the sum of divisors of the positive integer m, the ratio $\sigma(F_n)/F_n$ is dense in $[1, \infty)$. The present method now shows that there are infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\sigma(F_n)$ starts with the digits of s_2. Also, one may replace the Fibonacci sequence F_n in the above statements with some other sequence u_n for which it has been proved that $\phi(u_n)/u_n$ and $u_n/\sigma(u_n)$, respectively, are dense in $[0, 1]$. For example, one can take $u_n = 2^n - 1$ (see [1]) and the main result of this paper still holds provided that b is not a power of 2. We give no further details.

Acknowledgments. We thank the referee for a careful reading of the paper and for comments which improved its quality. This paper was written during a visit of F. L. to the Applied Mathematics Department of Naval Postgraduate

School in December, 2012. During the preparation of this paper, F. L. was supported in part by Project PAPIIT IN104512 (UNAM), VSP N62909-12-1-4046 (Department of the US Navy, ONR–Global) and a Marcos Moshinsky fellowship.

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Received January 10, 2012
Accepted June 19, 2013

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