EQUIDISTRIBUTION MOD $q$ OF ABUNDANT AND DEFICIENT NUMBERS

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ABSTRACT. The ancient Greeks called the natural number $m$ deficient, perfect, or abundant according to whether $\sigma(m) < 2m$, $\sigma(m) = 2m$, or $\sigma(m) > 2m$. In 1933, Davenport showed that all three of these sets make up a well-defined proportion of the positive integers. More precisely, if we let

$$ \mathcal{D}(u; x) := \left\{ m \leq x : \frac{m}{\sigma(m)} \leq u \right\}, \quad \text{and put} \quad D(u; x) := \# \mathcal{D}(u; x),$$

then Davenport’s theorem asserts that $\lim_{x \to \infty} \frac{1}{x} D(u; x)$ exists for every $u$. Moreover, $D(u)$ is a continuous function of $u$, with $D(0) = 0$ and $D(1) = 1$. In this note, we study the distribution of $\mathcal{D}(u; x)$ in arithmetic progressions. A simple to state consequence of our main result is the following: Fix $u \in (0, 1]$. Then the elements of $\mathcal{D}(u; x)$ approach equidistribution modulo prime numbers $q$ whenever $q$, $x$, and $\frac{x}{q \log \log \log x}$ all tend to infinity.

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1. Introduction

Recall that the natural number $m$ is said to be deficient if $\sigma(m) < 2m$ (for example, $m = 10$), perfect if $\sigma(m) = 2m$ (for example, $m = 6$), and abundant if $\sigma(m) > 2m$ (for example, $m = 12$). This classification goes back to the ancient Greeks; however, it was only in the 20th century that significant progress was made in understanding how these numbers were distributed within the sequence of natural numbers. For each $u \in [0, 1]$ and each real $x \geq 1$, put

$$ \mathcal{D}(u; x) := \left\{ m \leq x : \frac{m}{\sigma(m)} \leq u \right\}, \quad \text{and put} \quad D(u; x) := \# \mathcal{D}(u; x).$$

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In 1933, Davenport [4] showed that for all \( u \in [0, 1] \), the limit

\[
D(u) := \lim_{x \to \infty} \frac{1}{x} D(u; x)
\]

exists. Moreover, \( D(u) \) is a continuous function of \( u \), with \( D(0) = 0 \) and \( D(1) = 1 \). From these results, one quickly deduces that the deficient numbers have natural density \( 1 - D\left(\frac{1}{2}\right) \), that the perfect numbers have density 0 (here one uses the continuity of \( D(u) \)), and that the abundant numbers have density \( D\left(\frac{1}{2}\right) \). It is of some interest to obtain accurate numerical approximations of these values; improving on much earlier work, Kobayashi [10] has recently shown that the density of the abundant numbers lies between 0.24761 and 0.24766.

In 1946, Erdős [5] showed that the abundant and deficient numbers have the distribution predicted by Davenport’s result even in remarkably short intervals (see [5, Theorem 7(iii)]; see also [1] for closely related material). In fact, he showed that if \( A = A(x) \to \infty \), then

\[
\lim_{x \to \infty} \frac{\# \{ m \in (x, x + A \log x) : \frac{m}{\sigma(m)} \leq u \}}{A \log x} = D(u)
\]

for every fixed \( u \in [0, 1] \). The primary purpose of this note is to illustrate how Erdős’s ideas may be adapted to study the distribution of abundant and deficient numbers in arithmetic progressions. Specifically, we establish a sufficient condition for the elements of \( D(u; x) \) to approach equidistribution modulo \( q \), as both \( q \) and \( x \) tend to infinity. Our proof uses the same ideas that feature in Erdős’s work [5], supplemented by the method of moments.

It is certainly necessary to assume that \( x \to \infty \) to meaningfully discuss equidistribution, but why assume that \( q \to \infty \)? It turns out that for fixed \( q > 1 \) and \( u \in (0, 1) \), the elements of \( D(u; x) \) do not approach equidistribution as \( x \to \infty \). Let us quickly explain why. Consider those \( m \in D(u; x) \) which are 0 mod \( q \). Included here are all \( m \in [1, x] \) of the form \( nq \), where \( n/\sigma(n) \leq u \). The density of \( n \) satisfying \( n/\sigma(n) \leq u \) is \( D(u) \), which already implies that the limiting proportion of elements of \( D(u; x) \) that are 0 mod \( q \) is at least \( 1/q \). However, there are many \( m \) still unaccounted for! For instance, a positive proportion of natural numbers \( n \) are both coprime to \( q \) and satisfy \( u < n/\sigma(n) \leq u\sigma(q)/q \). (The proof of this parallels the proof that Davenport’s distribution function \( D \) is strictly increasing; compare with [13, Exercise 35, p. 275].) For these \( n \), the number \( m = nq \) also satisfies \( m/\sigma(m) \leq u \). It follows that the lower density of \( m \equiv 0 \pmod{q} \) satisfying \( m/\sigma(m) \leq u \) is strictly larger than \( \frac{1}{q} D(u) \), contradicting equidistribution. This analysis can easily be extended beyond the residue class 0 mod \( q \) to all of the residue classes \( a \pmod{q} \) with gcd \((a, q) > 1 \).
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Thus, equidistribution for fixed moduli $q$ is not in the cards. So to obtain equidistribution results, we must allow $q$ to vary with $x$. To avoid the difficulties discussed in the last paragraph, we also assume that $\sigma(q)/q = 1 + o(1)$.

**Definition.** Let $\mathcal{D}$ be an infinite set of natural numbers. We say that $\mathcal{D}$ is asymptotically tame if $\frac{\sigma(q)}{q} \to 1$ as $q \to \infty$ through elements of $\mathcal{D}$.

For instance, the prime numbers form an example of an asymptotically tame set. Our main theorem establishes equidistribution in a wide range of $q$ and $x$ provided that $q$ is restricted to an asymptotically tame set. We write $\log^k x$ for the $k$th iterate of the function $\log_1 x := \max\{1, \log x\}$.

1.1 Let $\mathcal{D}$ be an asymptotically tame set of natural numbers. Let $u$ be a fixed real number with $0 < u \leq 1$. For $q$ restricted to $\mathcal{D}$, we have that whenever $q, x$, and $\frac{x}{q \log_3 x}$ all tend to infinity,

$$\frac{\#\{m \leq x : m \equiv a \mod q, \frac{m}{\sigma(m)} \leq u\}}{\#\{m \leq x : m \equiv a \mod q\}} \to D(u),$$

uniformly in the choice of residue class $a$ mod $q$.

To see that this really is an equidistribution result, note that the denominator here is $\sim x/q$, while $D(u; x) \sim D(u)x$; thus, the proportion of elements of $\mathcal{D}(u; x)$ belonging to the progression $a \mod q$ is asymptotically $1/q$. As we explain after the proof of this theorem, the range of uniformity in $q$ is in some sense sharp.

The method of moments can also be used to establish several closely related results. Rather than try to formulate the most general theorem possible, we focus on a single theorem that is fairly representative of what may be expected.

Recall that every prime $p$ possesses $\varphi(p - 1)$ primitive roots (i.e., generators of the multiplicative group modulo $p$). Work of Burgess [3] shows that for $X := p^{\frac{1}{4} + \epsilon}$, the number of primitive roots mod $p$ in $[1, X]$ is asymptotic to $\frac{\varphi(p - 1)}{p^{\frac{1}{2}}} X$ as $p \to \infty$. Our second theorem shows that these small primitive roots also follow Davenport’s distribution.

**Theorem 1.2.** Fix $\epsilon > 0$ and fix $u \in (0, 1]$. As $p$ tends to infinity through prime values,

$$\frac{\#\{\text{primitive roots} 1 \leq m \leq p^{\frac{1}{4} + \epsilon} : \frac{m}{\sigma(m)} \leq u\}}{\#\{\text{primitive roots} 1 \leq m \leq p^{\frac{1}{4} + \epsilon}\}} \to D(u).$$

**Notation and conventions**

Throughout, we reserve the letter $p$ for a prime variable. We employ $O$ and $o$-notation, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with the usual meanings. We write $p^e \parallel m$ to mean that $p^e \mid m$ but that $p^{e+1} \nmid m$. 101
We use $\omega(m)$ for the number of distinct primes dividing $m$. Other notation will be introduced as necessary.

2. Preliminary remarks on the moments of $\frac{n}{\sigma(n)}$

The proofs of both theorems hinge on the following well-known result. See, for example, the textbook of Billingsley [2, Theorems 30.1 and 30.2, pp. 406–408].

**Lemma 2.1.** Let $F_1, F_2, F_3, \ldots$ be a sequence of distribution functions. Suppose that each $F_n$ corresponds to a probability measure on the real line concentrated on $[0, 1]$. For each $k = 1, 2, 3, \ldots$, assume that

$$
\mu_k := \lim_{n \to \infty} \int u^k \, dF_n(u)
$$

exists. Then there is a unique distribution function $F$ possessing the $\mu_k$ as its moments, and $F_n$ converges weakly to $F$.

In order to apply Lemma 2.1 we will need a convenient expression for the moments of Davenport’s distribution function $D$.

**Lemma 2.2.** Let $k$ be a natural number. The $k$th moment of $D(u)$ is given by the absolutely convergent sum

$$
\mu_k = \sum_{d_1, \ldots, d_k} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]}. \tag{2.1}
$$

Here the $d_i$ run over all natural numbers, and the multiplicative function $g$ is defined by the convolution identity

$$
\frac{n}{\sigma(n)} = \sum_{d \mid n} g(d).
$$

**Proof.** For each natural number $M$, let $D_M(u) := \frac{1}{M} \# \{m \leq M : \frac{m}{\sigma(m)} \leq u \}$. Then the distribution functions $D_M$ converge weakly to $D$; as a consequence,

$$
\int u^k \, dD(u) = \lim_{M \to \infty} \int u^k \, dD_M(u).
$$

(Compare with [2, Corollary, p. 348].)
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We now calculate
\[
\int u^k \, dD_M(u) = \frac{1}{M} \sum_{m \leq M} (m/\sigma(m))^k
\]
\[
= \frac{1}{M} \sum_{d_1, \ldots, d_k \leq M} g(d_1) \cdots g(d_k) \sum_{m \leq M} \frac{1}{\text{lcm}[d_1, \ldots, d_k]|m}.
\]

The final sum is $M/\text{lcm}[d_1, \ldots, d_k] + O(1)$, which shows that the $k$th moment of $D_M$ is given by
\[
\sum_{d_1, \ldots, d_k \leq M} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]} + O\left(\frac{1}{M} \sum_{d_1, \ldots, d_k \leq M} |g(d_1) \cdots g(d_k)|\right). \tag{2.2}
\]

On each prime power $p^e$, we find that
\[
g(p^e) = \frac{p^e}{\sigma(p^e)} - \frac{p^{e-1}}{\sigma(p^{e-1})} = -\frac{p^{e-1}}{\sigma(p^e)\sigma(p^{e-1})},
\]
and so in particular, $|g(p^e)| < 1/p^e$. Consequently, $|g(n)| \leq 1/n$ for every $n$.

Hence, the error term in (2.2) is $O(\frac{1}{M} (1 + \log M)^k)$, which vanishes as $M \to \infty$.

If we show that the sum (2.1) defining $\mu_k$ is absolutely convergent, then taking the limit as $M \to \infty$ in (2.2) will complete the proof of the lemma. But absolute convergence follows immediately from the bounds
\[
\left| \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]} \right| \leq \frac{1}{d_1 \cdots d_k \cdot \text{lcm}[d_1, \ldots, d_k]} \leq \prod_{i=1}^{k} \frac{1}{d_i^{1+1/k}},
\]
using for the final inequality that
\[
\text{lcm}[d_1, \ldots, d_k] \geq \max\{d_1, \ldots, d_k\} \geq (d_1 \cdots d_k)^{1/k}. \quad \square
\]

3. Proof of Theorem 1.1

The theorem is equivalent to the following proposition asserting weak convergence of certain distribution functions. Let $\mathcal{Q}$ be an asymptotically tame set, and let \{\(x_j\), \(q_j\), and \(a_j\)\} be sequences satisfying the following three conditions:

(i) each \(x_j \geq 1\), and \(x_j \to \infty\) as \(j \to \infty\),

(ii) each \(q_j \in \mathcal{Q}\), and both \(q_j\) and \(\frac{x_j}{q_j \log_3 x_j}\) tend to infinity,

(iii) each \(a_j \in \mathbb{Z}\).
If these conditions are satisfied, we write \( A_j := \frac{x_j}{q_j \log_3 x_j} \), so that \( A_j \to \infty \) as \( j \to \infty \).

For each \( j \), let \( D_j \) be the distribution function defined by
\[
D_j(u) := \frac{\# \{ m \leq x_j : m \equiv a_j \mod q_j \text{ and } \frac{m}{\sigma(m)} \leq u \}}{\# \{ m \leq x_j : m \equiv a_j \mod q_j \}}. \tag{3.1}
\]

**Proposition 3.1.** As \( j \to \infty \), \( D_j \) converges weakly to \( D \).

From Lemma 2.1, the proposition will follow if we can show that for each fixed \( k \),
\[
\lim_{j \to \infty} \int u^k \, dD_j(u) = \mu_k,
\]
with the \( \mu_k \) as defined in (2.1). To begin with, we compute that
\[
\int u^k \, dD_j(u) = \sum_{m \leq x_j} \left( \frac{m}{\sigma(m)} \right)^k \sum_{m \equiv a_j \mod q_j} 1. \tag{3.2}
\]
As \( j \to \infty \), the denominator in (3.2) is asymptotic to \( x_j/q_j \). We turn now to a study of the numerator. In what follows, we adopt the notation
\[
m' := \prod_{p \mid m, p \mid q_j} p^{e}, \quad m'' := \prod_{p \mid m, p \mid q_j, p \leq \log_3 x_j} p^{e}, \quad m''' := \prod_{p \mid m, p \mid q_j, p > \log_3 x_j} p^{e};
\]
clearly,
\[
m = m' m'' m'''.
\]
First we work on an upper bound. Since \( \frac{m}{\sigma(m)} \leq \frac{m''}{\sigma(m'')} \), we have
\[
\sum_{m \equiv a_j \mod q_j} \left( \frac{m}{\sigma(m)} \right)^k \leq \sum_{m \equiv a_j \mod q_j} \left( \frac{m''}{\sigma(m'')} \right)^k.
\]
Recalling the definition of the arithmetic function \( g \), we see that
\[
\sum_{m \equiv a_j \mod q_j} \left( \frac{m''}{\sigma(m'')} \right)^k = \sum_{d_1, \ldots, d_k \leq x_j} \sum_{p \mid d_1 \Rightarrow p \leq \log_3 x_j} g(d_1) \cdots g(d_k) \sum_{m \equiv a_j \mod q_j \gcd(d_i, q) = 1, \lcm[d_1, \ldots, d_k] \mid m} 1.
\]

The inner sum on the right-hand side is\[ \frac{x_j}{q_j} \sum_{d_1, \ldots, d_k \leq x_j} g(d_1) \cdots g(d_k) \frac{1}{\text{lcm}[d_1, \ldots, d_k]} + O(1), \] which shows that this last expression is (3.3)

\[ \frac{x_j}{q_j} \sum_{d_1, \ldots, d_k \leq x_j} g(d_1) \cdots g(d_k) \frac{1}{\text{lcm}[d_1, \ldots, d_k]} + O\left( \sum_{d_1, \ldots, d_k \leq x_j} |g(d_1) \cdots g(d_k)| \right). \]

To estimate the error, we recall that \(|g(d)| \leq 1/d|\), so that

\[ \left| \sum_{d_1, \ldots, d_k \leq x_j} g(d_1) \cdots g(d_k) \right| \leq \left( \sum_{d \geq 1} \frac{1}{d} \right)^k \]

\[ = \prod_{d \leq \log_3 x_j} \left( 1 - \frac{1}{p} \right)^{-k} \leq (2 \log_3 x_j)^k \]

once \(j\) is large. (We use Mertens’ theorem here as well as the bound \(e^\gamma < 2\).) Since \(x_j/q_j = A_j \log x_j\), where \(A_j \to \infty\), we see that the error term in (3.3) is \(o(x_j/q_j)\). Thus,

\[ \frac{1}{x_j/q_j} \sum_{m \leq x_j} \left( \frac{m''}{\sigma(m'')} \right)^k = \sum_{d_1, \ldots, d_k \leq x_j} g(d_1) \cdots g(d_k) \frac{1}{\text{lcm}[d_1, \ldots, d_k]} + o(1), \] (3.4)

as \(j \to \infty\). Referring back to (3.2) and remembering that the sum defining \(\mu_k\) in (2.1) converges absolutely, we deduce that

\[ \limsup_{j \to \infty} \int u^k \, dD_j(u) \leq \limsup_{j \to \infty} \sum_{d_1, \ldots, d_k \leq x_j} g(d_1) \cdots g(d_k) \frac{1}{\text{lcm}[d_1, \ldots, d_k]}. \] (3.5)

Next, we develop an analogous lower bound. Fix a small positive \(\epsilon\), say \(\epsilon \in (0, \frac{1}{2})\). We claim that the number of \(m \leq x_j\) in the progression \(a_j \mod q_j\) for which

\[ \frac{m''}{\sigma(m'')} < 1 - \epsilon \]

is \(o(x_j/q_j)\), as \(j \to \infty\). This claim will be deduced from an upper bound on the product of the terms \(\sigma(m'')/m''\). For each prime power \(p^e\), we have \(\sigma(p^e) = \frac{p^e}{p^e - 1}\).
Consequently,

\[
\prod_{m \leq x_j, \ m \equiv a_j \mod q_j} \frac{\sigma(m^m)}{m^m} \leq \prod_{m \leq x_j, \ m \equiv a_j \mod q_j} \prod_{p|m, \ p|q_j} \frac{p}{p-1} \\
= \exp \left( \sum_{p \leq x_j, \ p|q_j} \frac{1}{\log p} \sum_{m \leq x_j, \ m \equiv a_j \mod q_j} 1 \right). \tag{3.6}
\]

Note that

\[
\frac{\log p}{p-1} = \log \left( 1 + \frac{1}{p-1} \right) < \frac{1}{p-1} \leq \frac{2}{p}.
\]

For primes \( p \leq x_j/q_j \), the inner sum in (3.6) is at most \( 1 + \frac{x_j}{q_j} \leq 2 \frac{x_j}{q_j} \), and so the contribution to the double sum from these primes is at most

\[
2 \frac{x_j}{q_j} \sum_{p > x_j/q_j} \frac{1}{p} \sum_{p > \log_3 x_j} \frac{1}{p^2} < \frac{x_j}{q_j \log_3 x_j},
\]

once \( j \) is large (using partial summation and the prime number theorem in the final step). Now suppose that \( p \) is a prime not dividing \( q_j \) with \( p > x_j/q_j \). Then \( p \) can divide at most one integer \( m \leq x_j \) from the progression \( a_j \mod q_j \), since the difference between any two such \( m \) has the form \( q_j \ell \) with \( \ell \leq x_j/q_j \). So letting

\[
\Pi := \prod_{m \leq x_j, \ m \equiv a_j \mod q_j} m,
\]

we see that

\[
\sum_{p > x_j/q_j} \frac{\log p}{p-1} \sum_{m \leq x_j, \ m \equiv a_j \mod q_j} 1 \leq 2 \sum_{p|\Pi} \frac{1}{p}.
\]

Now \( \Pi \leq (x_j)^{1 + x_j/q_j} \leq x_j^{2x_j/q_j} \), and so by the prime number theorem,

\[
\Pi \leq \prod_{p \leq 4 \frac{x_j}{q_j} \log(x_j)} p.
\]

(We assume here, as we may, that \( j \) is large.) Thus, \( \omega(\Pi) \) is at most the total count of primes up to \( 4 \frac{x_j}{q_j} \log(x_j) \), and the sum of \( \frac{1}{p} \) taken over the primes dividing \( \Pi \) is bounded above by the corresponding sum over the primes up to \( 4 \frac{x_j}{q_j} \log(x_j) \). As a consequence,
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$$2 \sum_{\substack{p | \Pi \ p \leq \sqrt{q_j} \log(x_j)}} \frac{1}{p} \leq \sum_{\substack{p \leq \sqrt{q_j} \log(x_j)}} \frac{1}{p} \leq 2 \log \log(4x_j \log(x_j)) + O(1) \leq 2 \log_2(x_j/q_j) + 2 \log_3 x_j + O(1).$$

Collecting our estimates and referring back to (3.6), we find that

$$\prod_{m \leq x_j, m \equiv a_j \mod q_j} \frac{\sigma(m''')}{m'''} \ll \exp \left( \frac{x_j}{q_j \log_3 x_j} \right) \left( \log(x_j/q_j) \right)^2 \left( \log_2 x_j \right)^2.$$

On the other hand, whenever $$ \frac{m'''}{\sigma(m''')} < 1 - \epsilon, $$ we have $$ \frac{\sigma(m''')}{m'''} > 1 + \epsilon; $$ hence, the number of these $$ m \leq x $$ is at most

$$ \log \prod_{m \leq x_j, m \equiv a_j \mod q_j} \frac{\sigma(m''')}{m'''} \ll \epsilon + \frac{x_j}{q_j \log_3 x_j} + \log_2(x_j/q_j) + \log_3 x_j. $$

The first three terms on the right-hand side are clearly $o(x_j/q_j)$ as $j \to \infty$. The last one is also, since $x_j/q_j = A_j \log_3 x_j$, where $A_j \to \infty$. This proves the claim.

Now recalling the tameness assumption and the identity $$ \frac{\sigma(q_j)}{q_j} = \sum_{d | q_j} \frac{1}{d}, $$ we get that

$$ \frac{m'}{\sigma(m')} \geq \prod_{p | q_j} \left( 1 - \frac{1}{p} \right) \geq 1 - \sum_{p | q_j} \frac{1}{p} \geq 1 - \left( \frac{\sigma(q_j)}{q_j} - 1 \right) = 1 - o(1) $$

as $j \to \infty$, uniformly in $m$. In particular, once $j$ is large, we always have

$$ \frac{m'}{\sigma(m')} \geq 1 - \epsilon. $$

Using $*$ for a sum restricted to $m$ having $$ \frac{m'''}{\sigma(m''')} > 1 - \epsilon, $$ we deduce that for large $j$,

$$ \sum_{\substack{m \leq x_j \ m \equiv a_j \mod q_j \ m \equiv a_j \mod q_j}} \left( \frac{m}{\sigma(m)} \right)^k \geq \sum_{\substack{m \leq x_j \ m \equiv a_j \mod q_j \ m \equiv a_j \mod q_j}}^* \left( \frac{m'}{\sigma(m')} \right)^k \left( \frac{m''}{\sigma(m'')} \right)^k \left( \frac{m'''}{\sigma(m''')} \right)^k \geq (1 - \epsilon)^{2k} \sum_{\substack{m \leq x_j \ m \equiv a_j \mod q_j}} \left( \frac{m''}{\sigma(m'')} \right)^k. $$

The final restricted sum differs from the corresponding unrestricted sum by at most $o(x_j/q_j)$, since the restriction only removes $o(x_j/q_j)$ terms.
(by our earlier claim), each of which is nonnegative and at most 1. Thus,

\[
\liminf_{j \to \infty} \int u^k \, dD_j(u) = \liminf_{j \to \infty} \sum_{m \equiv a_j \mod q_j} \frac{m \sigma(m)^k}{x_j/q_j} \geq \sum_{m \leq x_j} \frac{(m''/\sigma(m''))^k}{x_j/q_j}.
\]

Since \(\epsilon\) may be taken arbitrarily small, the last inequality remains valid without the factor of \((1 - \epsilon)^{2k}\). Now referring back to (3.4), we find that

\[
\liminf_{j \to \infty} \int u^k \, dD_j(u) \geq \liminf_{j \to \infty} \sum_{d_1, \ldots, d_k} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]}.
\]

Comparing (3.5) and (3.7), we see that our proposition will be proved if we show

\[
\lim_{j \to \infty} \sum_{d_1, \ldots, d_k} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]} = \sum_{d_1, \ldots, d_k} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]}.
\]

Using once again that \(|g(d)| \leq 1/d\) for all \(d\), we see that

\[
\left| \sum_{d_1, \ldots, d_k} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]} \right| - \left| \sum_{d_1, \ldots, d_k, q_j} \frac{g(d_1) \cdots g(d_k)}{\text{lcm}[d_1, \ldots, d_k]} \right| \\
\leq \sum_{p|q_j} \sum_{d_1, \ldots, d_k} \frac{1}{d_1 \cdots d_k \cdot \text{lcm}[d_1, \ldots, d_k]}.
\]

Writing \(d_i = pd'_i\), the inner summand is at most

\[
\frac{1}{pd_1 \cdots d_{i-1}d_{i+1} \cdots d_k \cdot \text{lcm}[d_1, \ldots, d_{i-1}, d'_i, d_{i+1}, \ldots, d_k]},
\]

and so the triple sum is crudely bounded above by

\[
k \left( \sum_{p|q_j} \frac{1}{p} \right) \sum_{d_1, \ldots, d_k} \frac{1}{d_1 \cdots d_k \cdot \text{lcm}[d_1, \ldots, d_k]} \leq k \left( \frac{\sigma(q_j)}{q_j} - 1 \right) \sum_{d_1, \ldots, d_k} \frac{1}{(d_1 \cdots d_k)^{1+1/k}} = k \left( \frac{\sigma(q_j)}{q_j} - 1 \right) \zeta(1 + 1/k).
\]
But as \( j \to \infty \), the final expression tends to 0 by the tameness hypothesis. This completes the proof.

**Optimality**

We might wonder whether, instead of assuming that \( \frac{x}{q \log_3 x} \to \infty \), we can get by with the weaker assumption that \( \frac{x}{q \log_3 x} \) is sufficiently large. Equivalently, we might wonder whether there is a large absolute constant \( A \) so that Proposition 3.1 is true with condition (ii) replaced by

\[(ii') \text{ each } q_j \in \mathcal{D}, q_j \to \infty, \text{ and each } q_j \leq \frac{x_j}{A \log_3 x_j}.\]

But this is not so. To see this, it is enough to show that no matter how large \( A \) is taken, there is an asymptotically tame set \( \mathcal{D} \) and sequences \( \{x_j\}, \{q_j\}, \{a_j\} \) satisfying conditions (i), (ii'), and (iii) for which the corresponding distribution functions \( D_j \), as defined in (3.1), do not converge weakly to \( D \). In fact, we will show that these conditions do not even guarantee that the first moments of \( D_j \) approach the first moment of \( D \). The argument is closely analogous to one presented by Erdős in detail (see [5, p. 532]), and so we only outline it.

- We let \( \mathcal{D} \) be the set of natural numbers \( q \) whose smallest prime factor exceeds \( \frac{1}{10} \log q \). Each \( q \in \mathcal{D} \) satisfies

\[
1 \leq \frac{\sigma(q)}{q} \leq \exp \left( \sum_{p|q} \frac{1}{p-1} \right) \leq \exp \left( \frac{20 \omega(q)}{\log q} \right).
\]

Since the maximal order of \( \omega(q) \) is \( \log q / \log \log q \) (cf. [7, p. 471]), the set \( \mathcal{D} \) is asymptotically tame.

- Let \( \{x_j\} \) be a sequence that tends to infinity. We will also assume at various points that all of the \( x_j \) are sufficiently large (possibly depending on \( A \)). For each \( j \), let \( t = t_j = \lfloor \log_3 x_j \rfloor \), and choose squarefree numbers \( n_{j,1}, n_{j,2}, \ldots, n_{j,t-1} \), all supported on disjoint subsets of the primes in \( [\log_3 x_j, \frac{1}{10} \log x_j] \) and satisfying

\[
\frac{n_{j,i}}{\sigma(n_{j,i})} \leq e^{-9/10}.
\]

Since \( \sum_{\log_3 x_j \leq p \leq \frac{1}{10} \log x_j} \log \frac{p}{\sigma(p)} \sim -\log_3 x_j \), this is easily seen to be possible by employing a greedy construction.

- Choose \( q_j \) with

\[
\frac{x_j}{2A \log_3 x_j} < q_j < \frac{x_j}{A \log_3 x_j} \quad (3.8)
\]

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as a solution to the system of simultaneous congruences

\[ q_j \equiv 1 \mod \prod_{p \leq \log x_j \ p | n_j, i, \cdots n_j, t-1} p, \]

and \( i q_j + 1 \equiv 0 \mod n_j, i \) (for \( 1 \leq i < t_j \)).

(Note that \( i < \log x_j \), so that \( i \) is invertible modulo \( n_j, i \).) The Chinese remainder theorem lets us to do this, since \( \prod_{p \leq \frac{1}{10} \log x_j} p \leq x_j^{1/5} < x_j / (2A \log_3 x_j) \).

Each \( q_j \) has smallest prime factor \( \geq \frac{1}{10} \log x_j \) and \( \log q_j \), and so \( q_j \in \mathcal{Q} \).

- Finally, we choose each \( a_j = 1 \). This finishes the selection of the set \( \mathcal{Q} \) and the sequences \( \{x_j\}, \{q_j\}, \) and \( \{a_j\} \).
- Despite (i), (ii'), and (iii) all being satisfied, the first moments of the \( D_j \) turn out to be too small. To make this precise, let \( \Delta \) be the first moment of \( D_j \), so that

\[ \Delta = \prod_{p} \left( \sum_{e=1}^{\infty} g(p^e) / p^e \right) \approx 0.67. \]

Then we can show that

\[ \limsup_{j \to \infty} \frac{1}{x_j / q_j} \sum_{m \leq x_j \ \text{and} \ m \equiv 1 \mod q_j} \frac{m}{\sigma(m)} < \Delta. \quad (3.9) \]

Note that the left-hand side here is the \( \limsup \) of the first moments of the \( D_j \).

To see why \( (3.9) \) holds, observe that we have rigged the behavior of the first several terms of the sum. Indeed, all of the terms \( 1 < m < t_j q_j \) that appear have the form \( m = i q_j + 1 \) for some \( 1 \leq i < t_j \), and so \( m / \sigma(m) \leq e^{-9/10} \). Thus, these terms contribute at most \( e^{-9/10} t_j \) to the sum. We bound the contribution of the terms \( m > t_j q_j \) by replacing \( m / \sigma(m) \) with \( m'' / \sigma(m'') \) and mimicking the upper bound argument of the theorem. (Note that it was not important there that \( A \to \infty \).) We find that the remaining terms make a contribution to the sum of size at most \( (x_j / q_j - t_j) \Delta + o(x_j / q_j) \). Piecing everything together, we find that the left-hand side of \( (3.9) \) is at most

\[ \limsup_{j \to \infty} \left( e^{-9/10} x_j / q_j + \Delta \left( 1 - \frac{t_j}{x_j / q_j} \right) \right). \]

The expression inside the \( \limsup \) is a weighted average (convex combination) of \( e^{-9/10} \approx 0.41 \) and \( \Delta \approx 0.67 \); moreover, the coefficient of \( e^{-9/10} \) in
this convex combination is $\gg_A 1$, because of (3.8). This is enough to guarantee that the rigged terms skew the lim sup in (3.9) below $\Delta$.

4. Proof of Theorem 1.2

We begin by quoting the following version of Burgess’s character sum estimate, a proof of which can be found in the text of Iwaniec and Kowalski [9, pp. 327–329].

**Lemma 4.1.** Let $p$ be a prime, and let $\chi$ be a nontrivial Dirichlet character mod $p$. Let $M$ and $N$ be integers with $N > 0$, and let $r$ be a positive integer. Then

$$\sum_{M < n \leq M + N} \chi(n) \ll N^{1 - \frac{1}{r}} p^{\frac{r+1}{r^2}} (\log p)^\frac{1}{r}.$$  

Here the implied constant is absolute.

We will also need the following result expressing the characteristic function of the primitive roots modulo $p$ in terms of Dirichlet characters (see [3, Lemma 5]).

**Lemma 4.2.** Let $p$ be a prime number. For each integer $m$, let

$$\xi(m) = \frac{\varphi(p - 1)}{p - 1} \left( \chi_0(m) + \sum_{d \mid p - 1, d > 1} \frac{\mu(d)}{\varphi(d)} \sum_{\chi \text{ of exact order } d} \chi(m) \right),$$

where $\chi_0$ denotes the principal character mod $p$ and the sum on $\chi$ is over those characters of exact order $d$. Then $\xi(m) = 1$ if $m$ is a primitive root mod $p$, and $\xi(m) = 0$ otherwise.

We can now commence the proof of Theorem 1.2. For each prime $p$, let $X = p^{\frac{1}{2} + \varepsilon}$ and introduce the distribution function

$$D_p(u) := \frac{\# \{ \text{primitive roots } 1 \leq m \leq X : \frac{m}{\sigma(m)} \leq u \}}{\# \{ \text{primitive roots } 1 \leq m \leq X \}}.$$

We will show that for each fixed positive integer $k$, the $k$th moment of $D_p$ converges to the $k$th moment $\mu_k$ of Davenport’s distribution function $D$.  

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as \( p \to \infty \). We start by writing
\[
\int u^k \, dD_p(u) = \frac{1}{\#\{\text{primitive roots } 1 \leq m \leq X\}} \sum_{m \leq X} \left( \frac{m}{\sigma(m)} \right)^k.
\]
(4.1)

In [3], it is shown that the count of primitive roots in \([1, X]\) is asymptotic to \( \frac{\varphi(p-1)}{p-1} X \) as \( p \to \infty \), and so we focus our attention on the estimation of the sum in (4.1). Using \( \xi \) for the function defined in Lemma 4.2,
\[
\sum_{m \leq X} \left( \frac{m}{\sigma(m)} \right)^k = \sum_{m \leq X} \xi(m) \left( \frac{m}{\sigma(m)} \right)^k,
\]
which can be expanded as
\[
\frac{\varphi(p-1)}{p-1} \left( \sum_{m \leq X} \left( \frac{m}{\sigma(m)} \right)^k + \sum_{d \mid p-1, d>1} \mu(d) \sum_{\chi \text{ of order } d} \sum_{m \leq X} \chi(m) \left( \frac{m}{\sigma(m)} \right)^k \right).
\]
(4.2)

Applying Lemma 4.1 with \( r \) a parameter to be chosen momentarily, we get
\[
\sum_{m \leq X} \chi(m) \left( \frac{m}{\sigma(m)} \right)^k = \sum_{m \leq X} \chi(m) \left( \sum_{d \mid m} g(d) \right)^k
\]
\[
= \sum_{d_1, \ldots, d_k \leq X} g(d_1) \cdots g(d_k) \chi(\text{lcm}[d_1, \ldots, d_k]) \sum_{n \leq \text{lcm}[d_1, \ldots, d_k]} \chi(n)
\]
\[
\ll X^{1-\frac{1}{r}p^{-\frac{1}{4}r^2}} (\log p)^{1/r} \sum_{d_1, \ldots, d_k \leq X} \frac{|g(d_1)| \cdots |g(d_k)|}{\text{lcm}[d_1, \ldots, d_k]^{1-\frac{1}{r}}}.\]

We now assume that \( r \geq 2 \). Using that each \(|g(d_i)| \leq 1/d_i\) and that
\[
\text{lcm}[d_1, \ldots, d_k] \geq (d_1 \ldots d_k)^{1/k},
\]
we find that the remaining sum on the \( d_i \) is \( O_k(1) \). Since there are precisely \( \varphi(d) \) characters \( \chi \) of order \( d \), we deduce that
\[
\sum_{d \mid p-1} \sum_{d>1} \mu(d) \sum_{\chi \text{ of order } d} \sum_{m \leq X} \chi(m) \left( \frac{m}{\sigma(m)} \right)^k \ll_k X^{1-\frac{1}{r}p^{-\frac{1}{4}r^2}} (\log p)^{1/r} \sum_{d \mid p-1} |\mu(d)|
\]
\[
= (2^{\omega(p-1)} (\log p)^{1/r} p^{-\frac{1}{r} + \frac{1}{4r^2}}) X.
\]

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Now choosing $r := \max\{2, 1 + \lfloor (4\epsilon)^{-1} \rfloor\}$, we obtain after a quick computation that this last expression is $O_\epsilon(X^{1/\delta})$ for a certain $\delta = \delta(\epsilon) > 0$.

Moreover, $\sum_{m \leq X} (m/\sigma(m))^k \sim \mu_k X$ as $p \to \infty$. Removing the terms in this sum with $m$ divisible by $p$ (which appear only when $\epsilon \geq \frac{3}{4}$) changes the sum by $O(X/p)$, which is $o(X)$ as $p \to \infty$. Hence,

$$\sum_{m \leq X \atop p\nmid m} \left(\frac{m}{\sigma(m)}\right)^k \sim \mu_k X.$$

Piecing things together, we conclude that the initial sum in (4.2) is asymptotic to $\frac{\varphi(p-1)}{p} \mu_k X$ as $X \to \infty$. Combining this with our earlier estimate for the denominator in (4.1), we see that the $k$th moment of $D_p$ tends to $\mu_k$ as $p \to \infty$, as desired.

5. Concluding remarks

Narkiewicz has called a set of integers weakly equidistributed modulo $q$ if the elements of the set that are coprime to $q$ are uniformly distributed among the coprime residue classes modulo $q$. (See, for example, [12].) Suppose that $u \in (0, 1]$ is fixed. Then the elements of $\mathcal{D}(u; x)$ become weakly equidistributed modulo each fixed $q$, as $x \to \infty$. More precisely, for every fixed coprime residue class $a \mod q$,

$$\#\{m \leq x : m \equiv a \mod q, \frac{m}{\sigma(m)} \leq u\} \sim$$

$$\sim \frac{1}{\varphi(q)} \#\{m \leq x : \gcd(m, q) = 1, \frac{m}{\sigma(m)} \leq u\}, \quad (5.1)$$

as $x \to \infty$. The weaker version of this claim, where logarithmic density takes the place of natural density, is a consequence of [6, Lemma 1.17, p. 61]. A full proof of (5.1) can be obtained either through the method of moments or by the more concrete methods of [11]. In fact, the moments argument is used in [14, Lemma 2.2] to show that the limiting proportion of $m$ with $m/\sigma(m) \leq u$ from a fixed residue class $a \mod q$ is the same for all classes $a \mod q$ sharing the same value of $\gcd(a, q)$.

Somewhat frustratingly, none of the methods alluded to in the last paragraph seem well-suited to establishing an analogue of Theorem 1.1, i.e., showing that for fixed $u > 1$, the asymptotic relation (5.1) holds uniformly in a wide range of $q$. Some care will be necessary to formulate the right conjecture here; Iannucci [8] has shown that if $q$ is the product of the primes up to $(\log x)^{1/2+\epsilon}$, so that $q \approx \exp((\log x)^{1/2+\epsilon})$, then the interval $[1, x]$ contains no abundant numbers relatively prime to $q$. 

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