YET ANOTHER FOOTNOTE
TO THE LEAST NON ZERO DIGIT OF n!
IN BASE 12

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À la mémoire de Pierre LIARDET

ABSTRACT. We continue the study, initiated with Imre Ruzsa, of the last non zero digit \( \ell_{12}(n!) \) of \( n! \) in base 12, showing that for any \( a \in \{3, 4, 6, 8, 9\} \), the set of those integers \( n \) for which \( \ell_{12}(n!) = a \) is not 3-automatic.

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1. Introduction

We pursue the study, initiated in [5], prolonged in [4], of the sequence \( (\ell_{12}(n!))_n \), where \( \ell_b(n) \) denotes the last (or final) non zero digit of the integer \( n \) in base \( b \). In other words, if \( v_b(n) \) denotes the largest integer \( v \) such that \( b^v \) divides \( n \), then \( \ell_b(n) \) is the integer in \( \{1, 2, \ldots, b-1\} \) congruent to \( n/v_b(n) \) modulo \( b \).

We do not repeat the introduction of the previous papers, but just recall in short, that \( \ell_{10}(n!) \) is even as soon as \( n \geq 2 \) because \( v_2(n!) \) is larger than \( v_5(n!) \), and moreover the sequence \( (\ell_{12}(n!))_n \) is 5-automatic. In the case of the sequence \( (\ell_{12}(n!))_n \), the number \( v_4(n!) \) is usually larger than \( v_3(n!) \), but may be also smaller. A key point in the study is that the sequence \( (\ell_{12}(n!))_n \) coincides on a set of asymptotic density 1 with a 3-automatic sequence taking only the values 4 and 8, each with asymptotic density 1/2; in [5] some doubts were expressed on the fact that the sequence \( (\ell_{12}(n!))_n \) could be automatic itself. Our aim is to prove the following result

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1On lit en première page de cet article Communicated by Pierre Liardet, formule qui masque l’enthousiasme et la précision que Pierre mettait dans son activité éditoriale, comme dans toutes ses entreprises.
**Theorem 1.** For $a \in \{4, 8\}$ the sequence $\{n; \ell_{12}(n!) = a\}$ is not 3-automatic. For $a \in \{3, 6, 9\}$ the sequence $\{n; \ell_{12}(n!) = a\}$ is not automatic.

We first prove the second point, showing that each of the three considered sequences has no gap, which permits to apply a criterion of Minsky and Pappert for non automatic sequences, revisited by Cobham (Proposition 3 below). We then show that each of the sequences considered in the first point is a 3-automatic sequence, modified by a non automatic sequence.

**2. The tools for the proof of Theorem 1**

The following proposition recalls points established in the previous papers [5] and [4].

**Proposition 1.** Let $u_0u_1u_2 \ldots u_n \ldots$ be the fixed point, starting with a 4 of the substitution $4 \rightarrow 448884884$, $8 \rightarrow 884448448$. The following holds true

(i) If $v_4(n!) > v_3(n!)$, a relation which occurs on a set of asymptotic density 1, one has $\ell_{12}(n!) = u_n$.

Let $n$ be an integer divisible by 144 for which $v_3(n!) \geq v_4(n!) + 2$, then

(ii) for $k \in \{0, 1, 2, 3\}$, we have $\ell_{12}((n+k)!) \in \{3, 6, 9\}$,

(iii) for $a \in \{3, 6, 9\}$, there exists $k \in \{0, 2, 3, 7\}$ such that $\ell_{12}((n+k)!) = a$.

The second ingredient, already used in our previous papers is Legendre’s relation connecting $v_p(a)(n!)$ and the sum of the digits $s_p(n)$ of $n$ in the prime base $p$, namely

$$v_p(a)(n!) = \left\lfloor \frac{n - s_p(n)}{a(p-1)} \right\rfloor. \tag{1}$$

The third ingredient is a straightforward corollary of the main result of [9].

**Proposition 2.** There exists a constant $C$ such that, for $n \geq 25$ one has

$$s_2(n) + s_3(n) > \frac{\log \log n}{\log \log \log n + C} - 1. \tag{2}$$

Our last ingredient is a sufficient criterion of Minsky and Pappert ([2], Theorem 10) implying that a sequence is not automatic.

**Proposition 3.** An infinite sequence of integers $(a_n)_n$ with zero asymptotic density (i.e., $a_n/n$ tends to infinity) and no gap (i.e., $a_{n+1}/a_n$ tends to 1) is automatic in no base.
3. Proof of Theorem 1

We first prove that there are many integers \( n \) for which \( s_3(n) \) is small.

**Lemma 1.** Let \((h(n))_n\) be an increasing sequence of integers tending to infinity. The sequence \( \{n; s_3(n) \leq h(n)\} \) is not lacunary, in the sense that the sequence of the quotients of its consecutive terms tends to 1.

**Proof.** Without loss of generality we may assume that \( h(n) \) is an even number, say \( h(n) = 2g(n) \), and that \( h(n) \leq \log n/2 \log 3 \). We consider the set \( S_h \) of integers \( n \) for which \( h(n) \geq 4 \) and \( s_3(n) \leq h(n) \). Let \( n \) be in \( S_h \); we are going to prove that the interval

\[
\left(n, n \left(1 + 3^{1-g(n)}\right)\right]
\]

contains an element of \( S_h \), which is enough to prove the lemma.

- If \( s_3(n) \leq h(n) - 1 \), we have \( s_3(n+1) \leq s_3(n) + 1 \leq h(n) \leq h(n+1) \) and so \( (n+1) \) belongs to \( S_h \).

- If \( s_3(n) = h(n) = 2g(n) \), we let \( n = \eta_K 3^K + \eta_{K-1} 3^{K-1} + \cdots + \eta_0 \) be the proper representation of \( n \) in base 3, with \( \eta_K \neq 0 \) and \( K = \lfloor \log n/ \log 3 \rfloor \). Writing \( g \) for \( g(n) \), we have

\[
\eta_K + \eta_{K-1} + \cdots + \eta_{K-g+2} \leq 2(g-1) = h(n) - 2,
\]

and so we have

\[
\eta_0 + \eta_1 + \cdots + \eta_{K-g+1} \geq 2.
\]

- If \( \eta_{K-g+1} = 2 \), we consider the integer \( n + 3^{K-g+1} \); it is in the prescribed interval and satisfies, since there is a carry over,

\[
s_3(n + 3^{K-g+1}) \leq s_3(n) = h(n) \leq h(n + 3^{K-g+1}).
\]

- If \( \eta_{K-g+1} \leq 1 \), there exists \( \ell < K - g + 1 \) with \( \eta_\ell \geq 1 \); we then consider the integer \( n + 3^{K-g+1} + 3^\ell \), which is in the prescribed interval and satisfies

\[
s_3(n + 3^{K-g+1} + 3^\ell) \leq s_3(n) = h(n) \leq h(n + 3^{K-g+1} + 3^\ell).
\]

This proves Lemma 1. \( \square \)

In the sequel, we shall apply Lemma 1 with

\[
h(n) = 2 \left\lfloor \frac{\log \log n}{18 \log \log \log n} \right\rfloor
\]

and so the sequence

\[
S = \{n \geq 25; s_3(n) \leq h(n)\}
\]

is not lacunary. \( (3) \)
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Since the expansion of 144 in base 3 is 12100, for \( n \) in \( S \) and large enough, we have

\[
s_3(144n) \leq 4s_3(n) \leq 4h(n) \leq \frac{4 \log \log(144n)}{9 \log \log \log(144n)}.
\]

This, together with (2) implies that when \( n \) is in \( S \) and is large enough, we have

\[
s_2(144n) \geq s_3(144n) + 4.
\]

We now combine this last relation with Legendre’s formula (1) and get that for sufficiently large \( n \in S \), one has

\[
v_3((144n)!) \geq v_4((144n)!) + 2. \tag{4}
\]

For \( 1 \leq a \leq 11 \), we let \( \mathcal{L}_a = \{ n ; \ell_{12}(n!) = a \} \).

We start by considering the case when \( a \in \{3, 6, 9\} \). By Proposition 1 (i), the set \( \mathcal{L}_a \) has zero asymptotic density; by (4), Proposition 1 (iii) and (3), it is not lacunary; so, by Proposition 3 it is not automatic and the second part of Theorem 1 is settled.

We turn now our attention to \( \mathcal{L}_a \) for \( a \in \{4, 8\} \); we let \( \mathcal{U}_a = \{ n ; u_n = a \} \). By Proposition 1 (i), \( \mathcal{L}_a \subset \mathcal{U}_a \) and those two sets have both density 1/2 and coincide on a set of density 1/2. Another feature of the 3-automatic sequence \( \mathcal{U}_a \), easily seen on its definition, is that out of 4 consecutive integers, one at least belongs to it. By (1), Proposition 1 (ii) and (3), \( \mathcal{U}_a \setminus \mathcal{L}_a \) is a non lacunary infinite sequence of asymptotic density 0, and thus, is not automatic. This implies the first part of Theorem 1 and ends its proof.

We finally remark, that the non automaticity in all bases for the sequences \( \mathcal{L}_4 \) and \( \mathcal{L}_8 \) is still to be proved.

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REFERENCES

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