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ON A GOLAY-SHAPIRO-LIKE SEQUENCE

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Dedicated to the memory of Pierre Liardet

ABSTRACT. A recent paper by P. Lafrance, N. Rampersad, and R. Yee studies the sequence of occurrences of 10 as a scattered subsequence in the binary expansion of integers. They prove in particular that the summatory function of this sequence has the "root N" property, analogously to the summatory function of the Golay-Shapiro sequence. We prove here that the root N property does not hold if we twist the sequence by powers of a complex number of modulus one, hence showing a fundamental difference with the Golay-Shapiro sequence.

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1. Introduction

In the paper [7] the authors study a sequence $(i_n)_{n\geq 0}$ involving the number $inv_2(n)$ of inversions in the binary expansion of the integer n, i.e., the number of occurrences of 10 as a scattered subsequence of the binary representation of the integer n. More precisely, defining $i_n := (-1)^{inv_2(n)}$ they prove in particular the following result.

Theorem 1 (Theorem 2 in [7]). *There exists a bounded, continuous, nowhere differentiable, 1-periodic function G such that*

$$S(N) := \sum_{0\leq n\leq N} i_n = \sqrt{N}G(\log_4 N).$$

This shows that the behavior of the summatory function of sequence $(i_n)_{n\geq 0}$ is quite similar to the behavior of the summatory function of the Golay-Shapiro sequence (see [4, 3]). Recall that the ± 1 Golay-Shapiro sequence $(a_n)_{n\geq 0}$ is

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defined by $a_n = (-1)^{w_n}$, where w_n counts the number of possibly overlapping 11's in the binary expansion of the integer n. This sequence can also be defined by $a_0 = 1$, and for all $n \geq 0$, the recurrence relations $a_{2n} = a_n$ and $a_{2n+1} = (-1)^n a_n$ (see [2]). It is then natural to ask, as the authors of [7] do, whether the sequence $(i_n)_{n\geq 0}$ satisfies the fundamental "root N" property of the Golay-Shapiro sequence, namely

$$\sup_{\theta\in\mathbb{R}} \left| \sum_{0\leq n\leq N} a_n e^{2i\pi n\theta} \right| = O(\sqrt{N}).$$

This question is furthermore justified not only by the fact that $(a_n)_{n\geq 0}$ admits a "digital" representation as $(i_n)_{n\geq 0}$ does (namely $a_n = (-1)^{u_n}$, where u_n is the number of possibly overlapping 11's in the binary expansion of n), but also by the fact that many other "digital" sequences have the root N property (see [1]). The purpose of this paper is to prove that the sequence $(i_n)_{n\geq 0}$ does not satisfy the root N property.

Remark 1. The Golay-Shapiro sequence is also called the Rudin-Shapiro or the Shapiro-Rudin sequence. Since Rudin [8, p. 855] acknowledges Shapiro's priority of [9], and since [9] and [6] appeared the same year, the sequence should indeed be called the "Golay-Shapiro sequence". Note that the fact that this sequence appears in a somewhat disguised form in the paper of Golay [6] can be found in the article of Brillhart and Morton [5] where they write that Odlyzko pointed out to them [6, bottom of p. 469].

2. Preliminary results

First we recall a property of sequence $(i_n)_{n\geq 0}$ given in [7].

Proposition 1 (Proposition 1 of [7]). *The sequence $(i_n)_{n\geq 0}$ satisfies $i_0 = 1$ and the following recurrence relations: for all $n \geq 0$,*

$$i_{2n+1} = i_n, \qquad i_{4n} = i_n, \qquad i_{4n+2} = -i_{2n}.$$

This proposition implies the following result on the summatory function of $(i_n z^n)_{n\geq 0}$.

Proposition 2. *Let z be a complex number. Define the sum $T(N, z)$ by*

$$T(N, z) := \sum_{0\leq n\leq 2^N-1} z^n \begin{pmatrix} i_n \\ i_{2n} \end{pmatrix}.$$

Then we have

$$T(N+1,z) = \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} T\big(N, z^2\big).$$

Proof. Separating even and odd indices in $T(N+1,z)$ and using Proposition 1 yields

$$\begin{aligned} T(N+1,z) &= \sum_{0\le 2n\le 2^{N+1}-1} z^{2n}\begin{pmatrix} i_{2n} \\ i_{4n}\end{pmatrix} + \sum_{0\le 2n+1\le 2^{N+1}-1} z^{2n+1}\begin{pmatrix} i_{2n+1} \\ i_{4n+2}\end{pmatrix} \\ &= \sum_{0\le n\le 2^{N}-1} z^{2n}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} i_{n} \\ i_{2n}\end{pmatrix} + \sum_{0\le n\le 2^{N}-1} z^{2n+1}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} i_{n} \\ i_{2n}\end{pmatrix} \\ &= \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} T(N, z^2). \end{aligned}$$

□

It happens that a single common transformation gives a simpler form for all matrices

$$\begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix}.$$

Proposition 3. *Let i be a square root of -1 and P be the matrix defined by*

$$P := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Then

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \qquad \text{and} \qquad P^{-1}\begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix} P = \begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}.$$

Proof. Straightforward. □

Proposition 4. *Let $M(z) = \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix}$. Then we have*

$$\sum_{0\le n\le 2^{2N}-1} i_n z^n = \frac{1}{2}\big(c_N(1-i) + d_N(1+i)\big),$$

where

$$c_N = \prod_{0\le k\le N-1}\Big(z^{4^k} - i\Big)\Big(z^{2.4^k} + i\Big) \quad \text{and} \quad d_N = \prod_{0\le k\le N-1}\Big(z^{4^k} + i\Big)\Big(z^{2.4^k} - i\Big).$$

Proof. Using Proposition 2 we have

$$T(N+1,z) = M(z)T\left(N, z^2\right) = M(z)M\left(z^2\right)T\left(N-1, z^4\right) = \cdots$$

hence, starting from $T(N,z)$ yields

$$T(N,z) = M(z)M\left(z^2\right)\cdots M\left(z^{2^{N-1}}\right)T\left(0, z^{2^N}\right).$$

Replacing N by $2N$, defining $\widetilde{M}(z) := \begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}$, and using Proposition 3 gives

$$\begin{aligned} T(2N,z) &= \left(P\widetilde{M}(z)P^{-1}\right)\cdots\left(P\widetilde{M}\left(z^{2^{2N-1}}\right)P^{-1}\right)T\left(0, z^{2^{2N}}\right) \\ &= P\widetilde{M}(z)\cdots\widetilde{M}\left(z^{2^{2N-1}}\right)P^{-1}T\left(0, z^{2^{2N}}\right) \\ &= P\begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & z^{2^{2N-1}}-i \\ z^{2^{2N-1}}+i & 0 \end{pmatrix}P^{-1}T\left(0, z^{2^{2N}}\right). \end{aligned}$$

Grouping the matrices in the last equality pairwise and noting that

$$\begin{pmatrix} 0 & z-i \\ z+i & 0 \end{pmatrix}\begin{pmatrix} 0 & z^2-i \\ z^2+i & 0 \end{pmatrix} = \begin{pmatrix} (z-i)(z^2+i) & 0 \\ 0 & (z+i)(z^2-i) \end{pmatrix}$$

we obtain

$$T(2N,z) = P\begin{pmatrix} c_N & 0 \\ 0 & d_N \end{pmatrix}P^{-1}T\left(0, z^{2^{2N}}\right) \tag{1}$$

with

$$c_N = \prod_{0\le k\le N-1}\left(z^{4^k}-i\right)\left(z^{2.4^k}+i\right)$$

and

$$d_N = \prod_{0\le k\le N-1}\left(z^{4^k}+i\right)\left(z^{2.4^k}-i\right) \tag{2}$$

Since for any complex number Z we have

$$T(0,Z) = Z^0\begin{pmatrix} i_0 \\ i_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

Equality (1) can be rewritten as

$$T(2N,z) = \frac{1}{2}\begin{pmatrix} c_N(1-i)+d_N(1+i) \\ c_N(1+i)+d_N(1-i) \end{pmatrix}. \tag{3}$$

So that we have

$$\sum_{0\le n\le 2^{2N}-1} i_n z^n = \frac{1}{2}\big(c_N(1-i)+d_N(1+i)\big). \qquad \square$$

3. The main result

Now we state and prove our main theorem.

Theorem 2. *Define $j := e^{2i\pi/3}$. Let z be a complex number such that there exists an integer $r > 0$ with $z^{4^r} = j$. Then there exists a positive constant C (depending on z) such that, for all N large enough, the following inequality holds:*

$$\left| \sum_{0 \le n \le 2^{2N}-1} i_n z^n \right| \ge C(2+\sqrt{3})^N.$$

In particular, for such a complex number z, $\left| \sum_{0 \le k \le n-1} i_k z^k \right|$ is not $O(\sqrt{n})$.

Proof. If $z^{4^r} = j$, then, for all $d \ge r$, we clearly have

$$z^{4^d} = \left(z^{4^r}\right)^{4^{r-d}} = j$$

(note that $j^3 = 1$ hence $j^4 = j$). Thus, since

$$(j-i)\left(j^2+i\right) = 2-\sqrt{3} \quad \text{and} \quad (j+i)\left(j^2-i\right) = 2+\sqrt{3},$$

we have

$$\begin{aligned} c_N &= \prod_{0 \le k \le N-1} \left(z^{4^k} - i\right)\left(z^{2.4^k} + i\right) \\ &= \prod_{0 \le k \le r-1} \left(z^{4^k} - i\right)\left(z^{2.4^k} + i\right) \prod_{r \le k \le N-1} (j-i)\left(j^2+i\right) \\ &= A_z(2-\sqrt{3})^N \end{aligned}$$

for some constant

$$A_z = \prod_{0 \le k \le r-1} \left(z^{4^k} - i\right)\left(z^{2.4^k} + i\right) \left[(j-i)(j^2+i)\right]^{-1} \ne 0.$$

Similarly $d_n = B_z(2+\sqrt{3})^N$ for some nonzero constant B_z.

Finally, this gives

$$\begin{aligned} \left| \sum_{0 \le n \le 2^{2N}-1} i_n z^n \right| &= \frac{|d_N(1+i) + (1-i)c_N|}{2} \\ &\ge \frac{|(2+\sqrt{3})^N |B_z| - |A_z|(2-\sqrt{3})^N|}{\sqrt{2}} \\ &\ge C_z(2+\sqrt{3})^N \end{aligned}$$

for some positive constant C_z and N large enough. □

Remark 2. Actually Theorem 2 above shows that there exists a dense set of real numbers θ such that

$$\left| \sum_{0 \le n \le N-1} i_n e^{2i\pi n\theta} \right| \quad \text{is not} \quad O(\sqrt{N}).$$

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