PALINDROMIC CLOSURES AND THUE-MORSE SUBSTITUTION FOR MARKOFF NUMBERS

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ABSTRACT. We state a new formula to compute the Markoff numbers using iterated palindromic closure and the Thue-Morse substitution. The main theorem shows that for each Markoff number \(m\), there exists a word \(v \in \{a, b\}^*\) such that \(m - 2\) is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word \(av\). This construction gives a new recursive construction of the Markoff numbers by the lengths of the words involved in the palindromic closure. This construction interpolates between the Fibonacci numbers and the Pell numbers.

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1. Introduction

Markoff numbers are fascinating integers; the reader may use the recent book by Martin Aigner [A] for studying them. These numbers are related to number theory, hyperbolic geometry, continued fractions and Christoffel words [A, M1, M2, F, Re1, Re2]. Many great mathematicians have worked on these numbers and the famous uniqueness conjecture by Frobenius is still unsolved [B, M1, M2, F, C1, C2]. Markoff numbers are positive integers that appear in the solution of the Diophantine equation

\[x^2 + y^2 + z^2 = 3xyz.\]

The first Markoff numbers are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461,

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37666, 43261, 51641; they are listed in the Sloane Encyclopedia of Integer Sequences (sequence number A002559). One shows that if a Markoff triple \((x, z, y)\), that is, a triple satisfying the previous Diophantine equation, has maximum \(z\), then the triple gives birth to two others, which are \((x, 3xy - z, z)\) and \((z, 3zy - x, y)\) (see [A] Section 3.1). One can construct a binary tree using these computations, were each node is a Markoff triple (see [A]). The Frobenius conjecture asserts that each Markoff number is the maximum of a unique Markoff triple \((A, Re2)\). In the work of Markoff [M1, M2], one find implicitly combinatorics on words and construction of balanced sequences [CF, BS, V, BdLR] on the alphabet \(\{11, 22\}\). The Markoff numbers are also linked with approximation theory and continued fractions [BRS, B].

In this article, we find a new relation between Markoff numbers and combinatorics on words. The main theorem shows that for each Markoff number \(m\) there exists a word \(v \in \{a, b\}^*\) such that \(m - 2\) is equal to the length of the iterated palindromic closure of the iterated antipalindromic closure of the word \(av\).

The *iterated palindromic closure* (due to Aldo de Luca) is used in combinatorics on words in order to generate standard Sturmian words and central words [LL, J, BS]. One defines first the *palindromic closure* \(w^+\) of a word \(w\): it is the shortest palindrome having \(w\) as prefix (it exists and is unique). The iterated palindromic closure \(\operatorname{Pal}(u)\) is then defined recursively by \(\operatorname{Pal}(1) = 1\) (the empty word), and \(\operatorname{Pal}(va) = (\operatorname{Pal}(v)a)^+\) for any word \(v\) and any letter \(a\).

The *iterated antipalindromic closure* appears in the literature in order to construct antipalindromes and to generalize the iterated palindromic closure [LLDL, BPTV]. In fact, when the alphabet is binary, the iterated antipalindromic closure of a word \(u\) is obtained by applying the Thue-Morse substitution to the iterated palindromic closure of \(u\) [LLDL].

As an application of the main theorem, we give a new computation of Markoff numbers by a recursive construction on the lengths of the words involved in the iterated palindromic closure. The lengths of these words allow us to state a recursive formula using a *directive sequence* \(d = d_1d_2 \ldots d_j\) with \(d_i\) on the alphabet \(\{a, b\}\). One interesting property is to recover the usual Fibonacci recursive construction if \(d_j \neq d_{j-1} \neq d_{j-2}\) and the usual Pell recursive construction if \(d_j \neq d_{j-1} = d_{j-2}\) [C3, BRS].

Note that in the articles [F, P] we find two other decompositions of the Markoff numbers as sums of positive integers: using properties of continued fractions in the work of Frobenius and properties of snake graphs in the work of Propp et al. (see also [A]).
2. Iterated palindromic closures

In the sequel we work with the usual notations in combinatorics on words [BS]. Let $A$ be a finite alphabet.

The reversal of a word $x = x_1x_2\ldots x_n$ with $x_i \in A$ is the word $\bar{x} = x_nx_{n-1}\ldots x_1$.

A word $p$ is a palindrome if it is equal to its reversal (that is $p = \bar{p}$).

The length of a word $u = u_1u_2\ldots u_m$, where $u_i \in A$, is equal to $m$ and is denoted $|u|$.

The concatenation of two words $u = u_1u_2\ldots u_m$ and $v = v_1v_2\ldots v_n$ is the word of the length $m + n$ given by $u \cdot v = u_1u_2\ldots u_mv_1v_2\ldots v_n$.

In this article we use the palindromic closure, introduced by A l d o d e L u c a [dL] (more precisely, it is the right palindromic closure): the palindromic closure of a word $x$ is the shortest palindrome having $x$ as a prefix; it exists and is unique; it is denoted by $x(+)$. For example, if $x = ab$, then $x(+) = aba$.

It is known that $x(+) = x'y\bar{x}'$, where $x = x'y$ with $y$ the longest palindrome suffix of $x$. We consider the iterated palindromic closure (also introduced in [dL]), denoted by Pal$(d)$: it is a mapping from the free monoid on $A$ into itself, defined recursively by

$$\text{Pal}(d_1d_2\ldots d_n) = \text{Pal}(d_1d_2\ldots d_{n-1})d_n(+) \quad \text{for} \quad d_i \in A,$$

with the initial condition Pal$(1) = 1$, where 1 denotes the empty word. This mapping is injective and $w$ is called the directive word of Pal$(w)$. For example, Pal$(aba) = abaaba$: indeed,

$$\text{Pal}(a) = a \quad \text{and} \quad \text{Pal}(ab) = \left( \text{Pal}(a)\bar{b} \right)(+) = (ab)(+) = aba$$

and then

$$\text{Pal}(aba) = \left( \text{Pal}(ab)a \right)(+) = \left( \text{aba} \right)(+) = abaaba.$$

We also use the Thue-Morse substitution, denoted by $\theta = (ab, ba)$: it is an endomorphism of the free monoid $\{a, b\}^*$ that maps the letter $a$ to $ab$ and the letter $b$ to $ba$. 

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3. Main theorem

From now on, we work with the binary alphabet $\mathcal{A} = \{a, b\}$. We give a link between the computation of Markoff numbers and the length of words computed by iterated palindromic closure and Thue-Morse substitution:

**Theorem 1.** For each word $v \in \{a, b\}^*$, the number $|\text{Pal} \circ \theta \circ \text{Pal}(av)| + 2$ is a Markoff number $\neq 1, 2$. The mapping defined in this way from $\{a, b\}^*$ into the set of Markoff numbers different from $1, 2$ is surjective. Injectivity of this mapping is equivalent to the Frobenius conjecture.

**Remark.** If $v'$ is obtained from $v$ by interchanging $a$ and $b$, one finds that $\text{Pal} \circ \theta \circ \text{Pal}(av)$ and $\text{Pal} \circ \theta \circ \text{Pal}(bv')$ have the same length. In other words, since the roles of $a$ and $b$ are symmetric, starting the word with $b$ would give exactly symmetric words of the same length, so that we can consider the word $av$ without loss of generality.

**Examples.** The first Markoff numbers (not equal to 1 or 2) are 5, 13 and 29. The Markoff number $m = 5$ is given by $v = 1$: indeed, $\text{Pal}(a) = a$, thus $\theta \circ \text{Pal}(a) = ab$ and then $\text{Pal} \circ \theta \circ \text{Pal}(a) = aba$, which is of length 3.

The Markoff number $m = 13$ is given by $v = a$: indeed, $\text{Pal}(aa) = aa$, thus $\theta \circ \text{Pal}(aa) = abab$ and then $\text{Pal} \circ \theta \circ \text{Pal}(aa) = abaaababaaba$, which is of length 11.

The Markoff number $m = 29$ is given by $v = b$ indeed, $\text{Pal}(ab) = aba$, thus $\theta \circ \text{Pal}(aba) = ababaababa$, and then $\text{Pal} \circ \theta \circ \text{Pal}(ab) = ababaababaababaababaababa$, which is of length 27.

**Proof.** Define the monoid homomorphism $\mu$ from the free monoid $\{a, b\}^*$ into $\text{SL}_2(\mathbb{Z})$ by

$$
\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.
$$

It is known that $\mu(u)_{12}$ is a Markoff number for each word lower Christoffel word $u$, and that each Markoff number $m$ is equal to $\mu(u)_{12}$ for some lower Christoffel word $u$, see [BLRS, Th. 8.10]. Moreover, the uniqueness of $u$ is equivalent to the Frobenius conjecture.

If $m \neq 1, 2$, then $u \neq a, b$; in this case $u = apb$, and it is known that $p = \text{Pal}(v)$ for some word $v$ in $\{a, b\}^*$; moreover, the mapping $v \mapsto a\text{Pal}(v)b$ is a bijection from $\{a, b\}^*$ onto the set of proper lower Christoffel words (this well-known result follows for example from [BLRS, Corollary 3.1]).

Consider the monoid homomorphism $\alpha$ from the free monoid $\{a, b\}^*$ into $\text{SL}_2(\mathbb{Z})$ defined by

$$
\alpha(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
$$
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We have
\[ \alpha(ab) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \mu(a), \quad \alpha(aabb) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \mu(b). \]

Consider \( \phi = (ab, aabb) \). Then \( \mu = \alpha \phi \).

Note that for each word \( m, b \phi(m) = \psi(m)b \), where \( \psi = (ba, baab) = (ba, ab)G \) with \( G = (a, ab) \). Using [BdLR Corollary 3.2], we see that the length of the Christoffel word \( a \text{Pal}(w)b \) is equal to \( h + i + j + k \), where \( \alpha(w) = \begin{pmatrix} h & i \\ j & k \end{pmatrix} \).

The word \( w \) is defined as follows: we have \( \phi(u) = \phi(apb) = ab \phi(p) aabb \) and we define \( w = b \phi(p)a \). Thus \( \phi(u) = awabb \). Then we have
\[
\mu(u) = \alpha \phi(u) = \alpha(awabb) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & i \\ j & k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}
= \begin{pmatrix} * & h + i + j + k \\ * & * \end{pmatrix}
\]
and therefore \( m = \mu(u)_{12} = h + i + j + k = |a \text{Pal}(w)b| \).

Furthermore,
\[
w = b \phi(p)a = \psi(p)ba = ((ba, ab)G(p))((ba, ab)G(a)) = (ba, ab)G(pa).
\]

Since \( p = \text{Pal}(v) \), we obtain
\[
G(pa) = G(\text{Pal}(v)a) = \text{Pal}(av),
\]
by Justin’s formula [Be J]. Thus \( w = (ba, ab) \circ \text{Pal}(av) \). The computation of \( m \) gives
\[
m = 2 + |\text{Pal}(w)|
= 2 + |\text{Pal} \circ (ba, ab) \circ \text{Pal}(av)|
= 2 + |E \circ \text{Pal} \circ (ba, ab) \circ \text{Pal}(av)|
= 2 + |\text{Pal} \circ E \circ (ba, ab) \circ \text{Pal}(av)|
= 2 + |\text{Pal} \circ (ab, ba) \circ \text{Pal}(av)|
= 2 + |\text{Pal} \circ \theta \circ \text{Pal}(av)|.
\]

\[ \Box \]

A word \( z \) is an antipalindrome if it is equal to the exchange of its reversal (that is \( z = E(\tilde{z}) \)). For example, \( z = aababb \) is an antipalindrome because its reversal is \( \tilde{z} = bbabaa \) and the exchange gives \( E(\tilde{z}) = aababb \).

As for the palindromic case, we use the antipalindromic closure and the iterated antipalindromic closure which are defined in the work of de Luca and De Luca [dLDL]. The antipalindromic closure of a word \( x \) is the shortest antipalindrome having \( x \) as a prefix; it is denoted by \( x^\oplus \). For example, if \( x = ab \), then \( x^\oplus = ab \) because \( ab \) is already an antipalindrome and if \( x = aa \),
then \( x^\oplus = aabb \). The iterated antipalindromic closure noted \( \text{AntiPal}(d) \) is defined by the recursive formula
\[
\text{AntiPal}(d_1 d_2 \cdots d_n) = (\text{AntiPal}(d_1 d_2 \cdots d_{n-1}) d_n)^\oplus
\]
and the initial condition \( \text{AntiPal}(1) = 1 \). For example, \( \text{AntiPal}(aba) = abbaababbaab \); indeed, \( \text{AntiPal}(a) = ab \), thus \( \text{AntiPal}(ab) = (\text{AntiPal}(a)b)^\oplus = abbaab \) and then \( \text{AntiPal}(aba) = (\text{AntiPal}(ab)a)^\oplus = (ababab)^\oplus = abbaababbaab \).

We see that \( \text{AntiPal}(aba) = ab \cdot ba \cdot ab \cdot ab \cdot ba \cdot ab = (ab, ba) \circ \text{Pal}(aba) \).

This is a general fact, as shown in [dLDL] Theorem 7.6.

**Theorem 2** (De Luca, De Luca). Let \( v \) be a word on the alphabet \( A = \{a, b\} \) and \( \theta = (ab, ba) \) be the Thue-Morse substitution. Then
\[
\text{AntiPal}(v) = \theta \circ \text{Pal}(v).
\]

**Corollary 3.** For each word \( v \in \{a, b\}^* \), the number \( |\text{Pal} \circ \text{AntiPal}(av)| + 2 \) is a Markoff number \( \neq 1, 2 \). The mapping defined in this way from \( \{a, b\}^* \) into the set of Markoff numbers different from 1,2 is surjective. Injectivity of this mapping is equivalent to the Frobenius conjecture.

### 4. Computation of Markoff numbers

The previous corollary gives a new way to compute the Markoff numbers by using iterated antipalindromic closures and iterated palindromic closures. We now give a recursive formula for computing the Markoff numbers.

**Theorem 4.** Consider \( d = \text{AntiPal}(av) \) with \( v \in \{a, b\}^* \). We write \( d = d_1 d_2 \cdots d_{|d|} \) with \( d_i \in \{a, b\} \). We let \( L_0 = L_1 = 1 \) and \( L_2 = L_1 + L_0 = 2 \). For \( j \geq 3 \) we define recursively the \( L_j \):
\[
L_j = \begin{cases} 
L_{j-1} & \text{if } d_j = d_{j-1}, \\
L_{j-1} + L_{j-2} & \text{if } d_j \neq d_{j-1} \neq d_{j-2}, \\
L_{j-1} + L_{j-2} + L_{j-3} & \text{if } d_j \neq d_{j-1} = d_{j-2}.
\end{cases}
\]

Then the Markoff number \( m_v \) is given by
\[
m_v = 1 + \sum_{j=0}^{|d|} L_j.
\]

Consider the example \( v = ab \).

We have \( d = \text{AntiPal}(aab) = \theta(\text{Pal}(aab)) = \theta(aabaa) = abbaababab \) and then
\[
L_0 = 1; L_1 = 1; L_2 = L_1 + L_0 = 1 + 1 = 2; L_3 = L_2 + L_1 = 2 + 1 = 3
\]
(because \( d_3 = a \neq d_2 = b \neq d_1 = a \)); \( L_4 = L_3 + L_2 = 3 + 2 = 5 \)
For the prefix of length one of $d$ are the length of each $W_j$ and thus $L_j$.

Thus we have

We define

A more compact way of writing the $L_i$’s is to write $d$ and above each letter the $L_i$:

$$d_v = a \ b \ a \ b \ a \ b \ a \ b \ a \ b \ 1 \ 1 \ 2 \ 3 \ 5 \ 5 \ 13 \ 13 \ 31 \ 44 \ 75$$

Proof. To prove the theorem, we use Justin’s Formula \cite{J,Be} \begin{equation}
\text{Pal}(d' d'') = \psi_{d'}(d'') \cdot \text{Pal}(d')
\end{equation}

with $d'$ a word on $\{a,b\}^*$ and $d''$ a letter.

We recall that

$$\psi_{d'}(a) = \psi_{d'_1} \left( \psi_{d'_2} \left( \cdots \psi_{d'_{|d'|}}(a) \right) \right) \quad \text{with} \quad \psi_{a}(a) = a, \ \psi_{a}(b) = ab$$

($\psi_a$ was previously denoted $G$) and $\psi_b(a) = ba, \psi_b(b) = b$.

In our construction, we use $d = \text{AntiPal}(av)$ with $v \in \{a,b\}^*$ and we have to study $\text{Pal}(d) = \text{Pal}(d_1 d_2 \ldots d_{|d|-1} d_{|d|})$. By successive applications of Justin’s Formula we find

$$\text{Pal}(d_1 d_2 \ldots d_{|d|}) = \psi_{d_1 d_2 \ldots d_{|d|-1}}(d_{|d|}) \cdot \text{Pal}(d_1 d_2 \ldots d_{|d|-1})$$

$$\text{Pal}(d) = \text{Pal}(d_1 d_2 \ldots d_{|d|}) = \psi_{d_1 d_2 \ldots d_{|d|-1}}(d_{|d|}) \cdot \psi_{d_1 d_2}(d_3) \cdot \psi_{d_1}(d_2) \cdot d_1.$$

We define

$$W_j = \psi_{d_1 d_2 \ldots d_{j-1}}(d_j)$$

and

$$L'_j = |W_j| \quad \text{for} \quad j = 1, \ldots, |d|.$$
Now we compute the recursive formula for prefixes of length at least three of $d$. We have six cases to consider, indeed, we use the directive word $d = (ab, ba) \circ \text{Pal}(av)$ and thus $aaa$ and $bbb$ are forbidden words in the directive word $d$. It is sufficient to use Justin’s formula for the following prefixes of $d$:

$$d'aba, \ d'bba, \ d'bba, \ d'aab, \ d'baa \ \text{and} \ d'abb \ \text{with} \ d' \in \{a, b\}^*.$$  

The first case of the recursive formula is given by the prefixes of $d'$ of the form $d'baa$. We write $d = d_1d_2 \ldots d_{j-1}d_j = d'baa$ for a given $j$ and we are in the case $d_j = d_{j-1} = a$. Thus we have by the definition

$$W_j = \psi_{d'ba}(a) \ \text{and} \ W_{j-1} = \psi_{d'b}(a).$$

We have

$$W_j = \psi_{d'ba}(a) = \psi_{d'b}(\psi_a(a)) = \psi_{d'b}(a) = W_{j-1}, \ \text{thus} \ W_j = W_{j-1}.$$  

We find $L_j' = L_{j-1}'$ for $d_j = d_{j-1} = a$. Similarly, by exchanging the roles of $a$ and $b$ that is by considering the prefixes of the form $d'abb$ we find $W_j = W_{j-1}$; thus $L_j' = L_{j-1}'$ for $d_j = d_{j-1} = b$.

The second case of the recursive formula is given by the prefixes of $d$ of the form $d'aba$. We write $d_1d_2 \ldots d_{j-1}d_j = d'aba$ for a some $j$ and we are in the case $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. By the definition

$$W_j = \psi_{d'ab}(a) \ \text{and} \ W_{j-1} = \psi_{d'a}(b) \ \text{and} \ W_{j-2} = \psi_{d'}(a),$$

we have

$$W_j = \psi_{d'ab}(a) = \psi_{d'a}(\psi_b(a)) = \psi_{d'a}(ba) = \psi_{d'a}(b) \cdot \psi_{d'a}(a) = \psi_{d'a}(b) \cdot \psi_{d'}(\psi_a(a)) = \psi_{d'a}(b) \cdot \psi_{d'}(a) = W_{j-1} \cdot W_{j-2}.$$  

Thus we have

$$W_j = W_{j-1} \cdot W_{j-2} \ \text{and} \ L_j' = L_{j-1}' + L_{j-2}'$$

for $d_j = a \neq d_{j-1} = b \neq d_{j-2} = a$. Similarly, for the prefixes of the form $d'bab$ by exchanging the roles of $a$ and $b$ we have

$$W_j = \psi_{d'ba}(b) = W_{j-1}W_{j-2} \ \text{and} \ L_j' = L_{j-1}' + L_{j-2}'$$

for $d_j = b \neq d_{j-1} = a \neq d_{j-2} = b$.

The third case is given by the prefixes of $d$ of the form $d''bba$. As $bbb$ is forbidden in $d$, thus we write

$$d_1d_2 \ldots d_{j-1}d_j = d'abba \ \text{and we have} \ d_j = a \neq d_{j-1} = d_{j-2} = b.$$  

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By the definition

\[ W_j = \psi_{d'}abb(a), \quad W_{j-1} = \psi_{d'}ab(b), \quad W_{j-2} = \psi_{d'}a(b) \quad \text{and} \quad W_{j-3} = \psi_{d'}(a) \]

for a given \( j \) we have

\[ W_j = \psi_{d'}abb(a) = \psi_{d'}a \left( \psi_b(\psi_b(a)) \right) = \psi_{d'}a(ba) = \psi_{d'}a \left( \psi_{d'}a(ba) \right) \]

\[ = \psi_{d'}a(b) \cdot \psi_{d'}a(b) \cdot \psi_{d'}a(a) = \psi_{d'}ab(b) \cdot \psi_{d'}a(b) \cdot \psi_{d'}a(a) \]

\[ = \psi_{d'}ab(b) \cdot \psi_{d'}a(b) \cdot \psi_{d'}a(a) \]

\[ = W_{j-1} \cdot W_{j-2} \cdot W_{j-3} \]

Thus we have

\[ W_j = \psi_{d'}abb(a) = W_{j-1} \cdot W_{j-2} \cdot W_{j-3} \]

and thus

\[ L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for} \quad d_j = a \neq d_{j-1} = d_{j-2} = b, \]

And similarly, for the prefixes of the form \( d'baab \) we find

\[ W_j = \psi_{d'baa}(b) = W_{j-1} \cdot W_{j-2} \cdot W_{j-3} \]

and

\[ L'_j = L'_{j-1} + L'_{j-2} + L'_{j-3} \quad \text{for} \quad d_j = b \neq d_{j-1} = d_{j-2} = a. \]

Finally, we have to compute the Markoff numbers by using Corollary 3:

\[ m = |\text{Pal}(d)| + 2 \quad \text{with} \quad d = \text{AntiPal}(av). \]

Thus by Justin’s Formula

\[ m = \left| \psi_{d_1d_2\ldots d_{|d|-1}}(d_{|d|}) \psi_{d_1d_2}(d_3) \psi_{d_1}(d_2) \right| + 2 \]

\[ = |W_{|d|}| + |W_{|d|-1}| + \cdots + |W_2| + |W_1| + 2 \]

\[ = 2 + \sum_{j=1}^{|d|} L_j = 1 + \sum_{j=0}^{|d|} L_j. \]

Note that in the second case of the recursive formula, we have a Fibonacci recurrence

\[ L_j = L_{j-1} + L_{j-2}. \]

In the third case of the recursive formula we have

\[ L_j = L_{j-1} + L_{j-2} + L_{j-3} \quad \text{and} \quad d_{j-1} = d_{j-2}. \]

By application of the first case of the recursive formula we find

\[ L_{j-1} = L_{j-2} \]

and then a Pell recurrence

\[ L_j = 2L_{j-2} + L_{j-3}. \]
REFERENCES


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